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EXTRAPOLATION OF ASYMPTOTIC EXPANSIONS BY A MODIFIED AITKEN δ^2-FORMULA

by

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Abstract.

A modified Aitken formula permits iterated extrapolations to efficiently estimate s_∞ from s_n when an asymptotic expansion

$$s_n = s_\infty + n^{-k}(c_0 + c_1 n^{-1} + c_2 n^{-2} + \cdots)$$

holds for some (unknown) coefficients c_j. We study the truncation and irregular error and compare the method with other forms of extrapolation.

1. Introduction.

We consider accelerating the convergence of sequences $\{s_n\}_{n=1}^{\infty}$ satisfying

$$s_n = s_\infty + c_0 n^{-k} + O(n^{-k-1}) \quad , \quad k > 0 \tag{1.1}$$

where $c_0 \neq 0$ is unknown.

In order to study this problem, consider for a moment a continuous case. Let s(t) be a continuous function of t with an asymptotic expansion

$$s(t) = s_\infty + t^{-k}(c_0 + c_1 t^{-1} + c_2 t^{-2} + O(t^{-3})) \quad , \quad c_0 \neq 0 \, , \; k > 0 \, . \tag{1.2}$$

We assume that $s''(t) \neq 0$ and that termwise differentiation is legal. (Note that an expansion like $c_0 + c_1 u^{-2} + c_2 u^{-4} + \cdots$ can be reduced to this by the change of variables $t = u^2$.) Define a new function $s^*(t)$ by

$$s^*(t) = s(t) - p \frac{s'(t)^2}{s''(t)} \quad . \tag{1.3}$$

Now

$$s'(t)^2 = k^2 c_0^2 t^{-2k-2}\left(1 + 2\,\frac{k+1}{k}\,\frac{c_1}{c_0}\,t^{-1} + O(t^{-2})\right)$$

and

$$s''(t) = k(k+1)\, c_0\, t^{-k-2}\left(1 + \frac{k+2}{k}\,\frac{c_1}{c_0}\,t^{-1} + O(t^{-2})\right) \; .$$

Therefore

$$p\,\frac{s'(t)^2}{s''(t)} = p \bullet \; \& \; t^{-k}\left(c_0 + 2\,\frac{k+1}{k}\,c_1 t^{-1} + O(t^{-2})\right)\cdot\left(1 - \frac{k+2}{k}\,\frac{c_1}{c_0}\,t^{-1} + O(t^{-2})\right)$$

$$= p \bullet \frac{k}{k+1}\, t^{-k}(c_0 + c_1 t^{-1} + O(t^{-2})) \, . \tag{1.4}$$

It follows that $s^* = s_\infty + O(t^{-k-2})$ if we take $p = \frac{k+1}{k}$ in (1.3).

This result still holds if we replace derivatives by symmetric differences. Therefore in the discrete case we propose computing

$$s_n^* = s_n - \frac{k+1}{k} \frac{\Delta s_n \cdot \nabla s_n}{\Delta s_n - \nabla s_n} . \qquad (1.5)$$

(If $\Delta s_n = \nabla s_n$, just set $s_n^* = s_n$.)

We will show that if the asymptotic expansion

$$s_n = s_\infty + n^{-k}(c_0 + c_1 n^{-1} + c_2 n^{-2} + O(n^{-3})) , \quad c_0 \neq 0 , \ k > 0 \qquad (1.6)$$

holds, then the process (1.5) can be iterated, giving the " s-formula":

$$s_n^0 = s_n$$

$$s_n^{i+1} = s_n^i - \frac{k+2i+1}{k+2i} \frac{\Delta s_n^i \cdot \nabla s_n^i}{\Delta s_n^i - \nabla s_n^i} , \quad i = 0,1,2,\ldots. \qquad (1.7)$$

This iteration is illustrated in Figure 1.

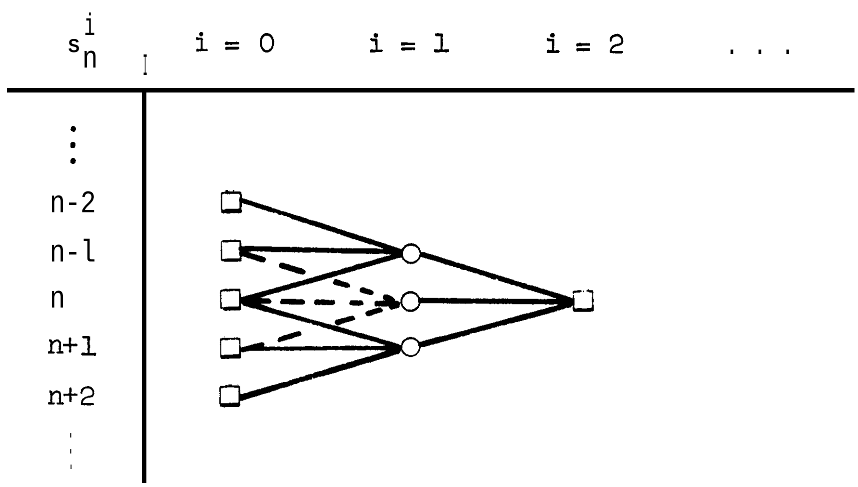

Figure 1. Iterated extrapolation process.

Note that the formula (1.3) can be used directly for the practical computation of s_∞ , if analytic differentiation is simple.

Often $a_n = s_n - s_{n-1}$ can be computed directly, with less rounding error than if s_n is first computed and then differenced. This occurs for example when

$$s_n = \sum_{j=1}^{n} a_j \tag{1.8}$$

and sometimes when s_n is computed using a recursion formula. In this case one should replace (1.5) by

$$s_n^* = s_n - \frac{k+1}{k} \frac{a_{n+1} a_n}{a_{n+1} - a_n} \quad . \tag{1.9}$$

If this is to be iterated one also needs a formula for the accurate computation of

$$a_n^* = s_n^* - s_{n-1}^* \quad . \tag{1.10}$$

Substituting (1.9) into (1.10) gives

$$a_n^* = a_n \left[\frac{k+1}{k} a_n \frac{\Delta a_n - \nabla a_n}{\Delta a_n \cdot \nabla a_n} - \frac{k+2}{k} \right] \tag{1.11}$$

This gives the mathematically equivalent "a-formula":

$$\begin{aligned} & a_n^0 = a_n \ , \quad s_n^0 = s_n \\ & s_n^{i+1} = s_n^i - \frac{k+2i+1}{k+2i} \cdot \frac{a_{n+1}^i a_n^i}{a_{n+1}^i - a_n^i} \ , \quad i = 0,1,2,\ldots \\ & a_n^{i+1} = a_n^i \left[\frac{k+2i+1}{k+2i} \cdot a_n^i \frac{\Delta a_n^i - \nabla a_n^i}{\Delta a_n^i \cdot \nabla a_n^i} - \frac{k+2i+2}{k+2i} \right] \ , \quad i = 0,1,2,\ldots \end{aligned} \tag{1.12}$$

(where again $s_n^{i+1} = s_n^i$ if $\Delta a_n^i = 0$, and $a_n^{i+1} = a_n^i$ if $\Delta a_n^i \cdot \nabla a_n^i = 0$).

Note that for greatest accuracy, s_n^i could be accumulated in the other direction .

In practice the error in $\Delta s_n^i - \nabla s_n^i$ and $\Delta a_n^i - \nabla a_n^i$ due to rounding and other irregular errors in s_n^i and a_n^i will limit the attainable accuracy. The irregular error component in s_n^{i+1} increases with n and i, while the truncation error decreases. We analyze this in the following two sections, then turn to a formula for estimating the exponent k which provides a check of assumptions and a possible termination criterion. After briefly discussing alternative techniques, we conclude with illustrations of the method.

2. Truncation Error.

We will now derive the leading term in the error expansion for $s_n^* - s_\infty$ assuming that an expansion of the form (1.6) holds for s_n. This establishes that the proposed method gains two orders for every application as indicated in Section 1.

$$s_n = s_\infty + n^{-k}(c_0 + c_1 n^{-1} + c_2 n^{-2} + O(n^{-3})) , \quad c_0 \neq 0 , \quad k > 0 . \qquad (2.1)$$

Using

$$(n \pm 1)^{-k} = n^{-k}(1 \mp kn^{-1} + \frac{k(k+1)}{2} n^{-2} \mp \ldots) \qquad (2.2)$$

we can expand Δs_n and ∇s_n in powers of n^{-1}.

$$\begin{aligned} \begin{matrix}\nabla s_n \\ \Delta s_n\end{matrix} = & - c_0 k n^{-k-1}(1 \pm \frac{k+1}{2} n^{-1} + \frac{(k+1)(k+2)}{3!} n^{-2} \pm \frac{(k+1)(k+2)(k+3)}{4!} n^{-3}) \\ & - c_1(k+1)n^{-k-2}(1 \pm \frac{k+2}{2} n^{-1} + \frac{(k+2)(k+3)}{3!} n^{-2}) \\ & - c_2(k+2)n^{-k-3}(1 \pm \frac{k+3}{2} n^{-1}) \\ & - c_3(k+3)n^{-k-4} + O(n^{-k-5}) . \end{aligned} \qquad (2.3)$$

Now

$$\begin{aligned} \Delta s_n \cdot \nabla s_n = & \; c_0^2 k^2 n^{-2k-2} + 2c_0 c_1 k(k+1)n^{-2k-3} \\ & + (c_0^2 k^2 \cdot \frac{k^2 + 6k + 5}{12} + 2c_0 c_2 k(k+2) + c_1^2(k+1)^2)n^{-2k-4} + O(n^{-2k-5}) \end{aligned} \qquad (2.4)$$

and

$$(\Delta s_n - \nabla s_n)^{-1} = \frac{n^{k+2}}{c_0 k(k+1)}\left(1 - \frac{c_1(k+2)}{c_0 k}\, n^{-1} + \left[\frac{k+2}{c_0 k}\left(\frac{c_1^2(k+2)}{c_0 k} - \frac{c_2(k+3)}{k+1}\right) - \frac{(k+2)(k+3)}{12}\right] n^{-2} + O(n^{-3})\right) . \quad (2.5)$$

Hence

$$\frac{k+1}{k}\,\frac{\Delta s_n \,.\, \nabla s_n}{\Delta s_n - \nabla s_n} = n^{-k}\left(c_0 + \frac{2c_1(k+1)}{k}\, n^{-1} + \left(c_0\,\frac{k^2+6k+5}{12} + \frac{2c_2(k+2)}{k} + \frac{c_1^2(k+1)^2}{c_0 k^2}\right) n^{-2} + O(n^{-3})\right)$$

$$\cdot \left(1 - \frac{c_1(k+2)}{c_0 k}\, n^{-1} + \left(-\frac{(k+2)(k+3)}{12} - \frac{c_2(k+2)(k+3)}{c_0 k(k+1)} + \frac{c_1^2(k+2)^2}{c_0^2 k^2}\right) n^{-2} + O(n^{-3})\right)$$

$$= n^{-k}\left(c_0 + c_1 n^{-1} + \left(\frac{c_0}{12}(k-1) + \frac{c_1^2}{c_0 k^2} + \frac{c_2}{k(k+1)}(k+2)(k-1)\right) n^{-2} + O(n^{-3})\right) .$$

This gives

$$s_n^* = s_\infty + \left(\frac{c_0}{12}(1-k) - \frac{c_1^2}{c_0 k^2} + \frac{2c_2}{k(k+1)}\right) n^{-k-2} + O(n^{-k-3}) . \quad (2.6)$$

By iterated use of (2.6) we obtain

$$s_n^i = s_\infty + c_0^i\, n^{-k-2i} + O(n^{-k-2i-1}) , \quad i = 0,1,2,\ldots \quad (2.7)$$

where $c_0^0 = c_0$, c_0^1 is given by (2.6) etc. The leading term in the truncation error is multiplied by $\frac{c_0^i}{c_0^{i-1}} \,.\, n^{-2}$ in iteration i , if $c_0^j \neq 0$, $j < i$, $i = 1,2,3,\ldots.$

By termwise differencing of the eqansion whose first term is given by (2.7) we get

$$a_n^i = \nabla s_n^i = -c_0^i(k+2i)n^{-k-2i-1} + O(n^{-k-2i-2}) \tag{2.8}$$

and therefore

$$\frac{a_n^{i+1}}{a_n^i} = \frac{c_0^{i+1}}{c_0^i} \cdot \frac{k+2i+2}{k+2i} \cdot n^{-2} + O(n^{-3}) \ , \ c_0^j \neq 0 \ , \ j \leq i \ . \tag{2.9}$$

3. The Propagation of Irregular Errors.

Usually the data s_n^0 , a_n^0 are subject to rounding errors or other errors in the process where they are computed or measured. There may also be transients in the sequence, which decay more rapidly than n^{-k} for large n but disturb the acceleration when n is not large enough, see Example 1. Such errors will be called irregular errors, since we do not assume that they depend smoothly on n , in contrast to the truncation error studied in Section 2. We shall study how the irregular errors in the given data are propagated in the acceleration process and derive approximate bounds that do depend smoothly on n .

Let $r(x)$, $r(y)$ be approximate bounds for the absolute value of the irregular error in the variables x, y . If a variable is defined by an arithmetic expression involving variables with previously known approximate error bounds, then the approximate error bound of the new variable is defined by the repeated application of the following fundamental rules:

$$r(x+y) \approx r(x) + r(y) \tag{3.1}$$

$$r(x-y) \approx r(x) + r(y) \tag{3.2}$$

$$r(x\cdot y) \approx |x|\cdot r(y) + |y|\cdot r(x) \tag{3.3}$$

$$r(x/y) \approx r(x)/|y| \;\text{[illegible]}\; r(y)/y^2 . \tag{3.4}$$

The sign of approximate equality means that the following simplifications are made:

First, when an error bound depends on n, we shall consider only the first term in its asymptotic expansion. Since usually $n \gg 1$, this is reasonable if one also assumes that the problem dependent coefficients c_j^i do not grow too fast with j and i.

Second, the use of the asymptotic estimates derived in Section 2 as well as the rules (3.3) and (3.4) are based on the assumption that all quantities are (much) larger than their irregular errors. In particular we assume that

$$|s_n^{i+1} - s_\infty| \ll |s_n^i - s_\infty| \tag{3.5}$$

where s_n^i, s_n^{i+1} denote quantities containing irregular errors. This assumption limits the number of extrapolations for which the theory can be applied. Fortunately this limit also indicates where the iterations, because of the propagated errors, do not any longer give significant improvements.

Third, we again emphasize that we study the propagation of the initial irregular errors only. The rounding errors committed in the accelleration process are assumed to be negligible. This is reasonable since in good floating point arithmetic, no new rounding errors are introduced when two almost equal numbers are subtracted. The inherited relative error of the result of such a subtraction will be much larger than the rounding errors in the later operations. Also the rounding errors in the final additions of the form $s_n^{i+1} = s_n^i + \eta_n$ where $|\eta_n| \ll |s_n^i|$, are negligible.

For convenient reference we collect here same formulas derived from the expansions in Section 2.

$$s_n^i - s_\infty = n^{-k-2i}(c_0^i + c_1^i n^{-1} + O(n^{-2})) \tag{3.6}$$

$$a_n^i = \nabla s_n^i = -c_0^i(k+2i)n^{-k-2i-1} + O(n^{-k-2i-2}) \tag{3.7}$$

$$a_{n+1}^i = \Delta s_n^i = a_n^i + O(n^{-k-2i-2}) \tag{3.8}$$

$$\Delta a_n^i = \Delta s_n^i - \nabla s_n^i = c_0^i(k+2i)(k+2i+1)n^{-k-2i-2} + O(n^{-k-2i-3})$$

$$= -(k+2i+1)a_n^i \cdot n^{-1} + O(n^{-k-2i-3}) \tag{3.9}$$

$$\Delta a_n^i - \nabla a_n^i = (k+2i+1)(k+2i+2)a_n^i\, n^{-2} + O(n^{-k-2i-4}) \tag{3.10}$$

In order to simplify the notation a bar over a variable will replace superscript $i+1$ and the superscript i will be omitted.

We first consider the s-formula.

$$\bar{s}_n = s_n - \frac{k+2i+1}{k+2i} \cdot \frac{\Delta s_n \cdot \nabla s_n}{\Delta s_n - \nabla s_n} . \tag{3.11}$$

Let

$$r(s_j) \approx \mu \ , \quad n-1 \le j \le n+1 \tag{3.12}$$

where we define μ to be an upper bound for $r(s_j)$. Using the rules (3.1) to (3.4) we get

$$r\left(\frac{\Delta s_n \cdot \nabla s_n}{\Delta s_n - \nabla s_n}\right)$$

$$\approx \frac{(|\nabla s_n|\cdot r(\Delta s_n) + |\Delta s_n|\cdot r(\nabla s_n)) \ . \ |\Delta s_n - \nabla s_n| + |\Delta s_n \cdot \nabla s_n| \cdot (r(\Delta s_n) + r(\nabla s_n))}{(\Delta s_n - \nabla s_n)^2}$$

$$\approx \frac{2\mu(|\Delta s_n| + |\nabla s_n|) \cdot |\Delta s_n - \nabla s_n| + 4\mu \cdot |\Delta s_n \cdot \nabla s_n|}{(\Delta s_n - \nabla s_n)^2}$$

$$\approx 4\mu \frac{|\Delta s_n \,.\, \nabla s_n|}{(\Delta s_n - \nabla s_n)^2} \qquad (3.13)$$

where in the last step we used relation (3.7) and (3.9) to drop higher order terms in n^{-1}. Using this; in (3.11) gives

$$r(\bar{s}_n) \approx \mu + 4\mu(k+2i+1) / (k+2i) \ . \ \frac{|\Delta s_n \nabla s_n|}{(\Delta s_n - \nabla s_n)^2}$$

$$\approx \frac{(2n)^2}{(k+2i)(k+2i+1)} \mu \qquad (3.14)$$

where we used (3.7),(3.8), and (3.9) to eliminate Δs_n , ∇s_n , and $\Delta s_n - \nabla s_n$.

Repeated application of (3.14) gives the final approximate upper bound on the error propagation.

$$r(s_n^i) \approx \frac{(2n)^{2i}}{k(k+1) \ . \ . \ . \ (k+2i-1)} \mu^0$$

where

$$\mu^0 = \max_{n-i \le j \le n+i} |r(s_j^0)|$$

and $r(s_j^0)$ is the initial error in element j of the original sequence $\{\cdot_n\}$.

Next consider the a-formula.

$$\bar{a}_n = \frac{k+2i+1}{k+2i} \cdot a_n^2 \frac{\Delta a_n - \nabla a_n}{\Delta a_n \nabla a_n} - (k+2i+2)/(k+2i) \cdot a_n \quad . \qquad (3.16)$$

Let

$$r(a_j) \approx \nu \quad , \quad n-1 \le j \le n+1 \; . \qquad (3.17)$$

where again ν is an upper bound for $r(a_j)$. We get

$$r(\bar{a}_n) \approx \frac{k+2i+1}{k+2i}\left(2r(a_n) \cdot \left|a_n \frac{\Delta a_n - \nabla a_n}{\Delta a_n \cdot \nabla a_n}\right| + a_n^2 \cdot r\left(\frac{\Delta a_n - \nabla a_n}{\Delta a_n \cdot \nabla a_n}\right)\right) + \frac{k+2i+2}{k+2i} \cdot r(a_n)$$

$$\approx \frac{k+2i+1}{k+2i}\left(2\nu \cdot \left|a_n \frac{\Delta a_n - \nabla a_n}{\Delta a_n \cdot \nabla a_n}\right| + 4a_n^2\left(\frac{\Delta a_n - \nabla a_n}{\Delta a_n \cdot \nabla a_n}\right)^2 \cdot \frac{|\Delta a_n \cdot \nabla a_n|}{(\Delta a_n - \nabla a_n)^2} \cdot \nu\right)$$

$$+ \frac{k+2i+2}{k+2i}\,\nu \qquad (3.18)$$

using (3.13) and (3.4) to find $r\left(\frac{\Delta a_n - \nabla a_n}{\Delta a_n \cdot \nabla a_n}\right)$.

Using (3. and (3.10) to eliminate Δa_n, ∇a_n, and $\Delta a_n - \nabla a_n$ in (3.18) we obtain

$$r(\bar{a}_n) \approx \frac{(2n)^2}{(k+2i)(k+2i+1)} \cdot \nu \quad . \qquad (3.19)$$

From (1.12) we have

$$r(\bar{s}_n) \approx r(s_n) + \frac{k+2i+1}{k+2i} \cdot \left[r \frac{a_{n+1}a_n}{\Delta a_n} \right]$$

$$\approx r(s_n) + \frac{k+2i+1}{k+2i} \cdot \left[\frac{|a_{n+1}| \cdot r(a_n) + |a_n| \cdot r(a_{n+1})}{|\Delta a_n|} + \frac{|a_{n+1}a_n| \cdot r(\Delta a_n)}{(\Delta a_n)^2} \right]$$

$$\approx r(s_n) + \frac{2n^2}{(k+2i)(k+2i+1)} \nu$$

$$\approx r(s_n) + \frac{1}{2} r(\bar{a}_n) \qquad (3.20)$$

again using (3.7),(3.8), and (3.9).

Repeated application of (3.20) yields finally

$$r(s_n^i) \approx r(s_n^0) + \frac{1}{2} \sum_{j=1}^{i} \frac{(2n)^{2j}}{k(k+1)\dots(k+2j-1)} \cdot \nu^0$$

$$\approx r(s_n^0) + \frac{1}{2} \frac{(2n)^{2i}}{k(k+1)\dots(k+2i-1)} \bullet \nu^0 \qquad (3.21)$$

where

$$\nu^0 = \max_{n-i \le j \le n+i} |r(a_j^0)|$$

and $r(a_j^0)$ is the initial error in element j of the original sequence $\{a_n\}$.

We conclude that the a-formula is never significantly inferior to the s-formula, and if

$$\nu^0 = \max_{n-i \le j \le n+i} |r(a_j^0)| \ll \max_{n-i < j < n+i} |r(s_j^0)| = \mu^0 \qquad (3.22)$$

the a-formula is indeed superior. We therefore feel that the a-formula (1.12) might as well be used even when a_n^0 has to be computed explicitly by $a_n^0 = s_n^0 - s_{n-1}^0$.

According to the analysis given above the irregular error in the extrapolated values will grow very fast with n and i . Since the truncation error decreases we should also consider the case in which the irregular error dominates the truncation error.

In Figure 2 and 3 we plot the truncation error, the irregular error component, the a priori bound on the irregular error given by (3.27), and the actual computed error using the a-formula in Example 2, Section 5. These quantities were obtained by doubling the precision of the original computation. We keep n+i constant and plot the above quantities on a $\log_{10}$ scale. The actual values for μ^0 and ν^0 (defined in (3.15) and (3.27)) are use-d. Note that the derivation of the bound given by (3.27) is valid only until the irregular error component reaches the level of the truncation error. The theoretical bound corresponds quite well to the actual behavior of the irregular error in the region where the assumptions for the analysis are valid.

As can be seen in Figure 2 and 3, and also from the examples in Section 5, the irregular errors level off in practice when they start dominating the truncation error. This phenomenon becomes plausible when we note that if Δs_n and ∇s_n were independent random variables ϵ_1 and ϵ_2 uniformly distributed on $(-\eta, \eta)$ then

$$P\left(\left|\frac{\epsilon_1 \epsilon_2}{\epsilon_1 - \epsilon_2}\right| < \frac{1}{2}\eta\right) > 0.7 \quad .$$

(This probability is relatively insensitive to the particular distribution chosen, as long as it is symmetric around 0.) The above result was found using 10,000 random trials.

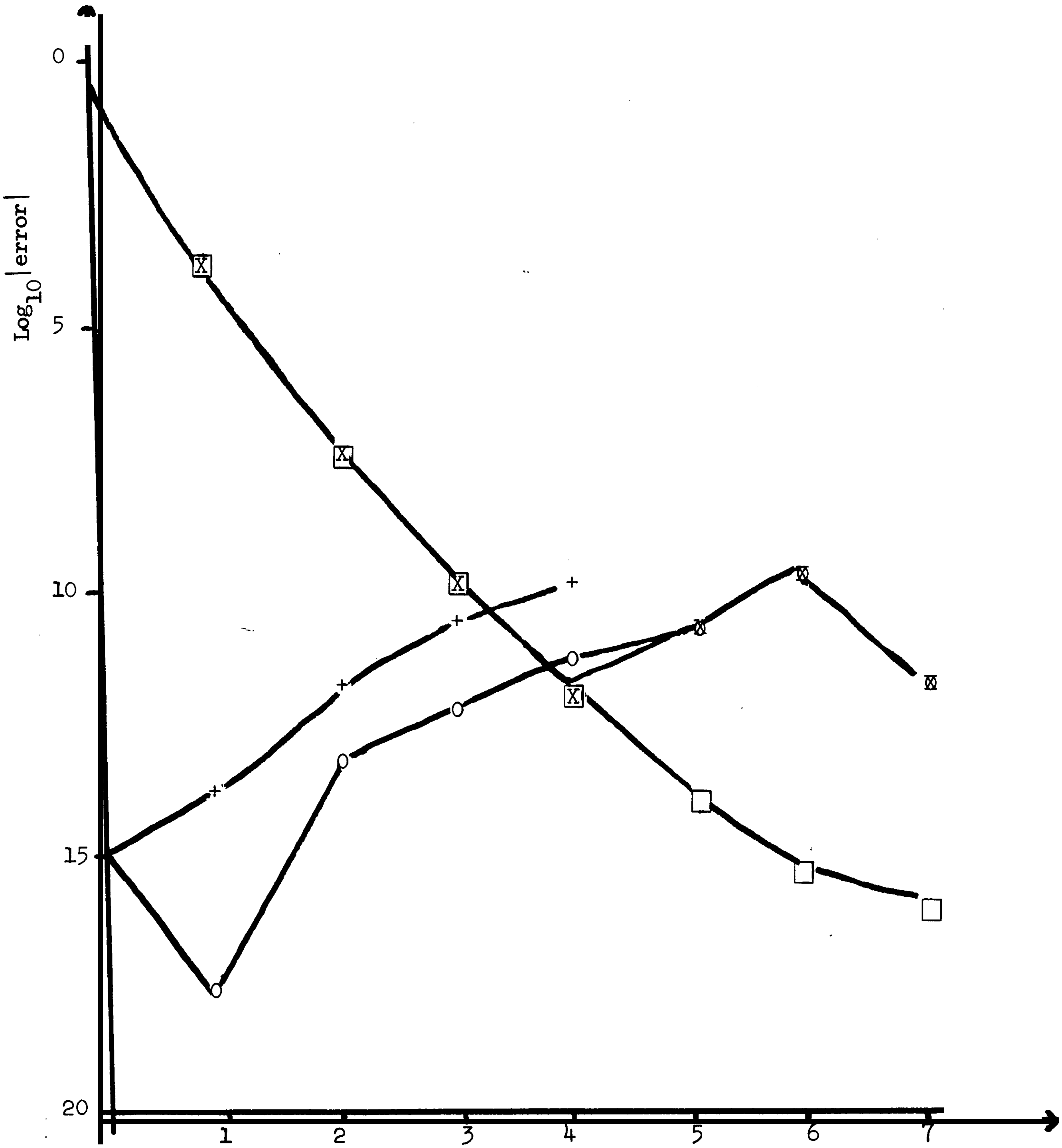

Figure 2. The a-formula used in Example 2, Section 5.

$n+i=15$ $\mu^0 = 9.9 \cdot 10^{-16}$. $\nabla^0 = 4.0 \cdot 10^{-17}$

□ Truncation error

0 Irregular error

X Actual error

\+ Expression 3.27.

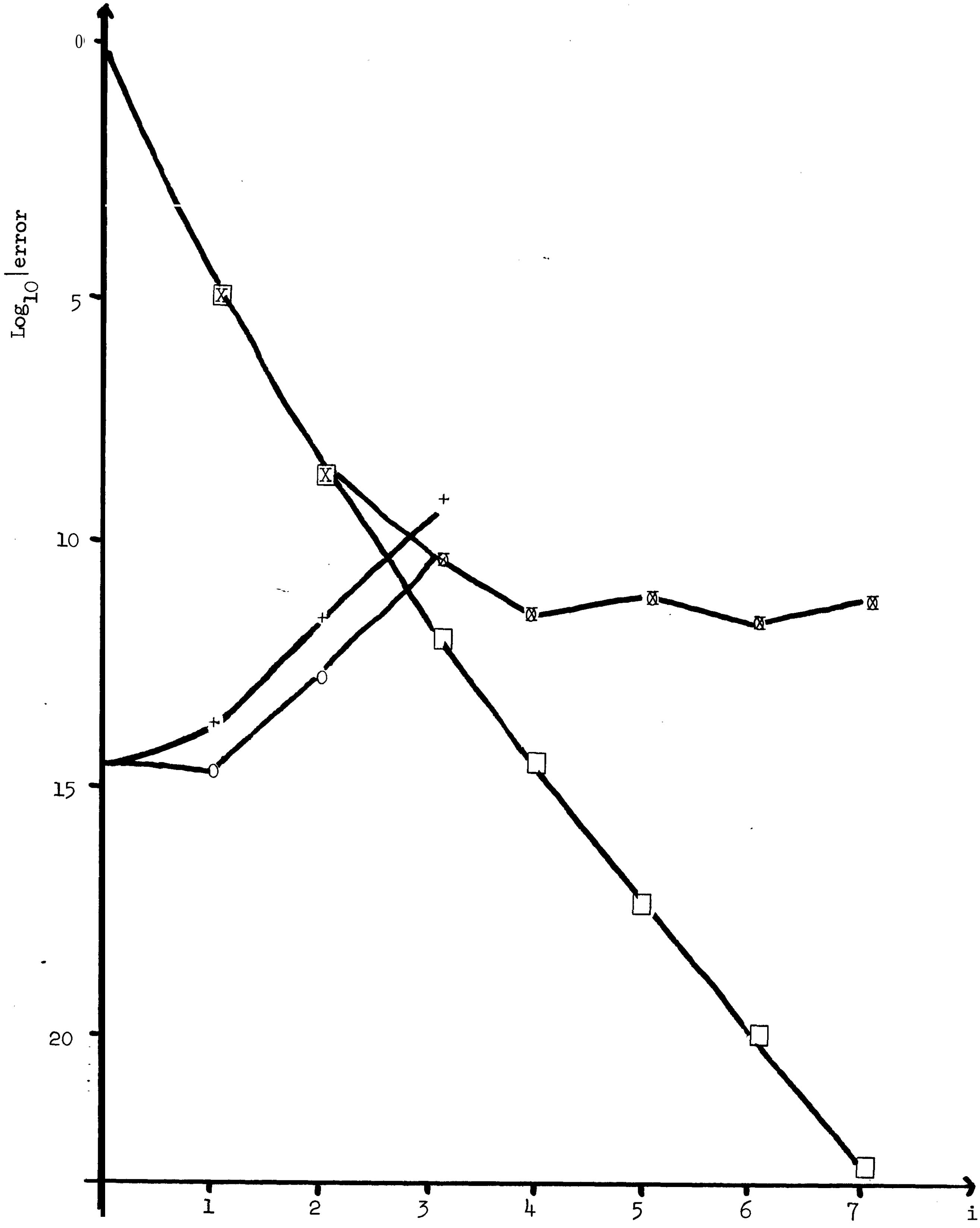

Figure 3. The a-formula used in Example 2. Section 5.
n+i=30. $\mu^0 = 7.6 \cdot 10^{-15}$. $\nabla^0 = 4.0$. 4.8

□ Truncation error   O Irregular error
X Actual error   + Expression 3.27.

4. Estimation of k , Termination Criterion.

Sometimes the coefficients c_j in the expansion (1.2) are so inaccessible that even verifying $c_0 \neq 0$ is difficult. In such cases an estimate of k directly from the data s_n is helpful.

Using the expansions (2.3) and (2.5) we find that for some constant c_*,

$$\frac{\Delta s_n}{\Delta s_n - \nabla s_n} = \frac{1}{c_0 k(k+1)} \Big[- n c_0 k + c_0 \frac{k(k+1)}{2} + c_1 + n^{-1}\left(- c_0 \frac{(k-1)\cdot k(k+2)}{12} + c_2 \frac{2k+4}{k+1} - \frac{c_1^2}{c_0} \frac{k+2}{k} \right) + c_* n^{-2} + O(n^{-3}) \Big] .$$

Differencing this gives

$$\Delta \left(\frac{\Delta s_n}{\Delta s_n - \nabla s_n} \right) = - \frac{1}{k+1} - n^{-2} \left[\frac{(k-1)(k+2)}{12(k+1)} - \frac{2c_2(k+2)}{c_0 k(k+1)^2} + \frac{c_1^2(k+2)}{c_0^2 k^2 (k+1)} \right] + O(n^{-3}) .$$

And therefore

$$k = -1 - \frac{1}{\Delta \left(\frac{\Delta s_n}{\Delta s_n - \nabla s_n} \right)} + O(n^{-2}) \qquad (4.1)$$

-Experiments indicate that symmetrized versions perform no better than this simple forward difference formula. (They will all have truncation error of order n^{-2} •) Note that a_{n+1} can replace Δs_n in (4.1).

This estimate may be applied in several ways. First, it provides a check of the underlying hypothesis; successive estimates k_n that steadily increase indicate an error expansion that may be exponential in -n , say, not polynomial in n^{-1} .

Next, the estimate suggest/the first nonzero term in the error expansion; sometimes because of problem symmetries this may not be obvious a priori.

Finally, it yields an appealing termination criterion. After each extrapolation, the estimate of k should increase by 2 , and once they depart from the expected behavior, the s_n^i from which they are derived have been sufficiently contaminated by irregular error that further extrapolation is pointless. For a somewhat similar heuristic in adaptive quadrature, see de Boor (1971).

Another more specific termination criterion which is easy to implement together with the recommended a-formula, can very naturally be based on the expected monotonicity of the sequences $\{s_n^i\}$. (For each i).

Suppose we are computing the sequence $\{s_n^i\}$ from the previous column $\{s_n^{i-1}\}_{n=N_{min}}^{N_{max}}$. (Or in practice, updating the corrections to the original sequence $\{s_n^0\}$ using the a-formula (1.12).) Then:

1. Find the index $N_1 \geq N_{min}$ for which the elements a_n^i , $n \geq N_1$, starts having constant sign. (Say, at least three consecutive elements, after a possible irregular sign pattern in the beginning.) Set $N_{min} = N_1$.

2. Find the first index $N_2 > N_1$ for which $a_{N_2} a_{N_2+1} < 0$.

   A. If $N_2 = N_{max}$ (that is, no sign change occured) let $s_*^i = s_{N_2}^i$ be the estimate of s_∞ from $\{s_n^i\}$ and take $TOL^{(\cdot)} = |a_{N_2}|$ to be an estimate (order of magnitude only!) of the error $|s_*^i - s_\infty|$.

B. If $N_2 < N_{max}$, let $N_{max} = N_2$, take $s_*^i = s_{N_2}^i$ and take

$TOL^{(i)} = \max\{|a_{N_2}|, |a_{N_2+1}|\}$.

3. If $TOL^{(i)} > TOL^{(i-1)}$, accept the estimate s_*^{i-1} with error estimate $TOL^{(i-1)}$, otherwise compute a new extrapolated column $\{s_n^{i+1}\}_{n=N_{min}}^{N_{max}}$

A FORTRAN implementation of this criterion running on an IBM 370/168 (precision approximately 16 decimals) gave

$$s_\infty \approx 0.13533528320 \pm 1.5 \times 10^{-11}$$

$$e^{-2} = 0.13533528324 \quad \text{(exact value)}$$

based on $\{s_n^0\}_{n=10}^{50}$ from Example 1 in Section 5.

while

$$s_\infty \approx 2.612375348 69 \pm 2.0 \times 10^{-11}$$

$$\zeta(1.5) = 2.61237534869 \quad \text{(exact value)}$$

based on $\{s_n^0\}_{n=1}^{15}$ from Example 2 in Section 5.

5. Alternative Methods.

From (1.1) we can immediately derive an "elimination formula".

$$t_n^* = t_{n+1} + \frac{\Delta t_n}{\left(\frac{n+1}{n}\right)^k - 1} \tag{5.1}$$

This formula, too, may be iterated to take full advantage of the asymptotic expansion (1.2). (See for example, Dahlquist and Björck (1974).) Note that two iterations of the above formula correspond to one iteration of the s-formula; both reduce the order of the truncation error by n^{-2}, using 3 data points. However, this method has two serious flaws. First it is not translation invariant with respect to n. As illustrated in the examples below, the proper choice of origin is not always obvious in practice. Second, the rounding errors do not level off, as they do for the s-formula. This means that a very careful termination criterion must be used if this formula were to be applied.

Another alternative is to use some other symmetric difference formula based on the continuous formula (1.3). For example

$$s_n^* = s_n - \frac{k+1}{k} \cdot \frac{\frac{1}{4}(\Delta s_n + \nabla s_n)^2}{\Delta s_n - \nabla s_n} \tag{5.2}$$

has truncation error

$$s_n^* = s_\infty + \left(-\frac{c_0}{12}(3k^2 + 7k + 2) - \frac{c_1^2}{c_0 k^2} + \frac{2c_2}{k(k+1)} \right) n^{-k-2} + O(n^{-k-3}) \tag{5.3}$$

which is comparable to the corresponding error (2.6) for the s-formula. (Note however that the factor multiplying c_0 is larger in (5.3).) Which of the two formulas will do better depends on the problem.

One nice feature of the s-formula (1.4) is that it can be interpreted as a modification of the classical Aitken extrapolation formula

$$s_n^{**} = s_n - \frac{(\Delta s_n)^2}{\Delta^2 s_n} \tag{5.4}$$

which assumes an error expansion

$$s_{n+1} - s_\infty = (A + \varepsilon_n)(s_n - s_\infty) , \quad |A| < 1 , \varepsilon_n \to 0 , n \to \infty . \tag{5.5}$$

(see, for example, Henrici (1964).)

This follows since in the limit $k = \infty$,

$$s_n^* - s_{n-1}^{**} = \Delta s_{n-1} - \frac{\Delta s_n \Delta s_{n-1} - (\Delta s_{n-1})^2}{\Delta^2 s_{n-1}} = \Delta s_{n-1} - \frac{\Delta s_{n-1} \Delta^2 s_{n-1}}{\Delta^2 s_{n-1}} = 0.$$

The equivalence also holds of course, for the formula

$$s_n^* = s_n - \frac{k+1}{k} \frac{(\Delta s_n)^2}{\Delta^2 s_n} . \tag{5.6}$$

This formula is not symmetric and will therefore only accelerate the convergence by one order of magnitude. ($s_n^* = s_\infty + O(n^{-k-1})$.) The last formula can, however, be useful in a quite different context when computing multiple roots of nonlinear equations. (See Overholt (1965).)

It should be noted that we assume the work required to evaluate s_n to increase substantially with n . If this is not the case, and any s_n can be accurately computed, better accuracy can be obtained by using a subsequence of s_n in the extrapolation process. For example one can evaluate s_n only for $n = mk$ or $n = 2^k$, $k = 1,2,3\ldots$, m some integer. The s - or a-formula can still be applied, setting $k = \infty$ in cases like

$n = 2^k$ (Aitken extrapolation). In this case the elimination method will be translation invariant and both extrapolation processes tend to be more stable; the irregular error component will not blow up.

In the special case $k = 1$, other extrapolation methods are available; see Joyce (1971) for a comprehensive catalog. Like the elimination scheme above, these other methods depend explicitly on the independent variable n, or more accurately on $h_n = 1/n$. Polynomial extrapolation,

$$p_n^0 = s(h_n)$$

$$p_n^{j+1} = p_{n+1}^j + \frac{p_{n+1}^j - p_n^j}{\frac{h_n}{h_{n+j+1}} - 1} \quad , \quad 0 \le j \; ,$$

uses Neville's iterative linear interpolation to evaluate at 0 the polynomial passing through the data. For a more complete treatment with applications to summation of series see Gander (1973). Rational extrapolation (Burlisch and Stoer (1964)),

$$r_n^{-1} = 0$$

$$r_n^0 = s(h_n)$$

$$r_n^{j+1} = r_{n+1}^j + \frac{r_{n+1}^j - r_n^j}{\left(\frac{r_n^j - r_{n+1}^{j-1}}{r_{n+1}^j - r_{n+1}^{j-1}}\right)\left(\frac{h_n}{h_{n+j+1}}\right) - 1} \quad , \quad 0 \le j \; ,$$

interpolates instead by a rational function. An alternate way to do this, based on Thiele's iterative reciprocal differences interpolation, is the ρ-algorithm proposed by Wynn (1956), (see also Brezinski (1977), Warner (1974)):

$$\rho_n^{-1} = 0$$

$$\rho_n^{0} = s(h_n)$$

$$\rho_n^{j+1} = \rho_{n+1}^{j-1} + \frac{\frac{1}{h_{n+j+1}} - \frac{1}{h_n}}{\rho_{n+1}^{j} - \rho_n^{j}}$$

Collapsing two steps yields

$$\rho_n^{j+2} = \rho_{n+1}^{j} + \frac{N}{D}$$

where

$$N = \left(\frac{1}{h_{n+j+2}} - \frac{1}{h_n}\right) \Delta \rho_{n+1}^{j} \nabla \rho_{n+1}^{j}$$

$$D = \Delta\rho_{n+1}^{j-1} \Delta\rho_{n+1}^{j} \nabla\rho_{n+1}^{j} + \left(\frac{1}{h_{n+j+2}} - \frac{1}{h_{n+1}}\right)\nabla\rho_{n+1}^{j} - \left(\frac{1}{h_{n+j+1}} - \frac{1}{h_n}\right) \Delta\rho_{n+1}^{j}$$

Wynn notes that as convergence progresses, the first term in the denominator can be neglected. Substituting $h_n = \frac{1}{n}$ then gives the s-formula.

From this is appears that when both apply, the s-formula and the p-algorithm (or, equivalently, rational extrapolation) will behave comparably, though not quite identically. The numerical experiments presented in the next section bear this out.

On the other hand, the range of applicability of the two methods is different. The s-formula allows general k but requires $h_n = \frac{1}{n}$, while rational extrapolation allows general h_n but requires $k=1$. Another difference is that the a-formula is able to take advantage of the accurate series terms a_n .

We have been unable to devise a satisfactory scheme generalizing both methods. Of course, elimination can still be applied; the case of general k and general h_n is handled by

$$q_n^0 = s(h_n)$$

$$q_n^{i+1} = \frac{\sum_{-1< m < 1} d_m \rho_{n+m}^i}{\sum_{-1< m < 1} d_m} ,$$

$$d_{-1} = (h_n - h_{n+1}) / h_{n-1}^{k+i}$$

$$d_0 = (h_{n+1} - h_{n\ 1}) / h_n^{k+1}$$

$$d_1 = (h_{n-1} - h_n) / h_{n+1}^{k+i}$$

However, limited numerical experiences suggests that this may not perform particularly well in practice for general h_n . (For $h_n = \frac{1}{n}$, the first elimination scheme and this one behave comparably.)

Polynomial extrapolation can be used with general h_n and k [Bauer, Rutishauser and Stiefel (1963)], but the trick depends on linearity of the extrapolation process, and therefore does not seem to carry over to rational extrapolation.

Another possibility, which we have not investigated, would be to fit the data by a least squares method. Perhaps stepwise regression based on the asymptotic expansion would be appropriate.

We conclude that there is still room for work on extrapolation methods, particularly in theoretically explaining empirical behavior such as the superiority of the s-formula and rational extrapolation to elimination methods.

6. Examples.

Example 1.

At the most recent Gatlinburg conference on linear algebra, nametags were scattered about the dining room to encourage people to make new acquaintances. One evening a participant remarked: "This can't be random -- I just sat next to this guy at lunch." Suppose that n people are seated randomly around a circular table for two meals. What is the probability p_n that no one can make such a remark'?

From a recurrence formula by Poulet (1919), we may compute that

$$p_2 = p_3 = p_4 = 0$$

$$p_5 = \frac{1}{12}$$

$$p_6 = \frac{1}{20}$$

$$p_7 = \frac{23}{360}$$

$$\begin{aligned} p_n = -[& (-n^6 + 17n^5 - 116n^4 + 415n^3 - 849n^2 + 978n - 504)p_{n-1} \\ & + (-4n^4 + 48n^3 - 188n^2 + 240n)p_{n-2} \\ & + (2n^4 - 30n^3 + 154n^2 - 300n + 144)p_{n-3} \\ & + (-n^2 + 7n - 9)p_{n-4} \\ & + (-n^2 + 5n - 3)p_{n-5}] \\ & /(n^6 - 17n^5 + 114n^4 - 385n^3 + 689n^2 - 618n + 216) \end{aligned}$$

$$\text{for } n \geq 8 . \tag{6.1}$$

(In particular, $p_{100} \approx 0.130$. In fact, the seating was not completely random, but the one incident would not be enough to establish this.)

The recurrence may be rewritten as

$$
\begin{aligned}
p_n - p_{n-1} = -[&(-2n^4 + 30n^3 - 160n^2 + 360n - 288)p_{n-1} \\
&+ (-4n^4 + 48n^3 - 188n^2 + 240n)p_{n-2} \\
&+ (2n^4 - 30n^3 + 154n^2 - 300n + 144)p_{n-3} \\
&+ (-n^2 + 7n - 9)p_{n-4} \\
&+ (-n^2 + 5n - 3)p_{n-5}] \\
&/(n^6 - 17n^5 + 114n^4 - 385n^3 + 689n^2 - 618n + 216). \quad (6.2)
\end{aligned}
$$

Since $0 \le p_n \le 1$, and $p_n - p_{n-1} = O(n^{-2})$, we see that $\lim p_n$ exists and an asymptotic expansion of the form (1.2) holds with k = 1 .

We might guess that in the limit the probability of success overall is just the product of the probability of success at n-l seats, so that $\lim_{n\to\infty} p_n = \lim_{n\to\infty}\left(\frac{n-3}{n-1}\right)^{n-1} = e^{-2} = 0.135335\ldots.$ It appears that no one has rigorously proven this yet, but we can easily check the conjecture numerically using the techniques described above. On the following pages we present the results of different extrapolation techniques applied to $s_n^{(0)} = p_n$. For clarity the predicted limit e^{-2} has been subtracted off the extrapolated values, and only every second elimination step is shown. For more detail see the 'program listings in the appendix. Estimates (4.1) of k based on the s-formula are included in the numerical results.

n	s_n^0	s_n^1	s_n^2	s_n^3	s_n^4	s_n^5
1	-135.3 (-03)					
2	-135.3 (-03)	-135.3 (-03)				
3	-135.3 (-03)	-135.3 C-03)	-135.3 (-03)			
4	-135.3 (-03)	-135.3 (-03)	-135.3 (-03)	-135.3 (-03)		
5	-52.00 (-03)	-99.62 (-03)	705.8 (-03)	271.6 (-03)	54.41 (-03)	
6	-85.34 (-03)	-65.73 (-03)	-16.49 (-03)	-as.00 (-03)	-262.3 (-03)	-157.5 (-03)
7	-71.45 (-03)	-48.05 (-03)	-69.79 (-03)	-193.1 (-03)	-12A.0 (-03)	-422.0 (-03)
8	-65.10 (-03)	-258.4 (-03)	-104.9 C-03)	-72.73 (-03)	144.6 (-03)	40.11 (-03)
9	-58.30 (-03)	-4.047 (-03)	8.465 (-03)	977●3 (-06)	1.062 (-03)	946.9 (-06)
10	-52.87 (-03)	5.003 (-03)	1.862 (-03)	1.051 (-03)	958.4 (-06)	1.069 (-03)
11	-48.29 (-03)	1.818 (-03)	1.249 (-03)	1.879 (-03)	3.699 (-03)	2.498 (-03)
12	-44.42 (-03)	1.442 (-03)	4.906 (-03)	2.423 (-03)	1.913 (-03)	36.16 (-03)
13	-41.11 (-03)	1.002 (-03)	141.6 (-06)	-59.10 (-06)	16.02 (-06)	1.991 (-06)
14	-38.25 (-03)	740.I (-06)	-19.97 (-06)	9.156 (-06)	3.567 (-06)	2.219 (-06)
15	-35.76 (-03)	560.8 (-06)	8.591 (-06)	3.888 (-06)	2.455 (-06)	3.161 (-06)
16	-33.56 (-03)	435.7 (-06)	4.049 (-06)	2.876 (-06)	3.935 (-06)	622.3 (-09)
17	-31.62 (-03)	34s.s (-06)	3.244 (-06)	13.87 (-06)	6.874 (-06)	4.572 (-06)
18	-29.89 (-03)	278.7 (-06)	2.359 (-06)	52.91 t-091	-151.5 (-09)	-53.77 (-09)
19	-28.34 (-03)	228.2 (-06)	1.753 (-06)	-123.7 (-09)	-62.43 (-09)	30.97 (-09)
20	-26.94 (-03)	189.3 (-06)	1.317 (-06)	-46.74 (-09)	-17.35 (-09)	29.49 (-09)
21	-25.67 (-03)	158.R (-06)	1.001 (-06)	-27.47 (-09)	4.430 (-09)	-467.2 (-12)
22	-24.51 (-03)	134.5 (-06)	170.6 (-09)	-16.07 (-09)	-1.096 (-09)	131.4 (-12)
23	-23.413 t-03)	115.0 (-06)	600.0 (-09)	-9.973 (-09)	284.5 (-12)	-540.0 (-12)
24	-22.48 (-03)	99.09 (-06)	472.3 (-09)	-6.343 (-09)	-1.321 (-09)	-16.38 t-12)
25	-21.59 (-03)	AS.99 (-06)	375.5 (-09)	-4.355 (-09)	3.048 (-09)	652.3 (-12)
26	-20.76 (-03)	75.12 (-06)	301.4 (-09)	-2.834 (-09)	-1.210 (-09)	2.562 (-09)
27	-20.00 (-03)	66.01 (-06)	244.t (-09)	-2.099 (-09)	15.52 (-09)	6.324 (-09)
28	-19.29 (-03)	58.32 (-06)	199.3 (-09)	-1.398 (-09)	-064.4 (-12)	-2.892 (-09)
29	-18.63 (-03)	51.78 (-06)	163.9 (-09)	-1.118 (-09)	-2.506 (-09)	-1.466 (-09)
30	-18.01 (-03)	46.19 (-06)	135.8 (-09)	-753.4 (-12)	-329.2 (-12)	-517.3 (-12)
31	-17.43 (-03)	41.39 (-06)	113.3 (-09)	-569.5 (-12)	-512.7 f-12)	-633.6 (-12)
32	-16.89 (-03)	37.21 (-06)	95.04 (-09)	-530●4 (-12)	-581.0 (-12)	-521.8 (-12)
33	-16.38 (-03)	33.59 (-06)	80.22 (-09)	-196.0 (-12)	-337.7 (-12)	-271.3 (-12)
34	-15.90 (-03)	30.42 (-06)	68.08 (-09)	-392.9 (-12)	-209.6 (-12)	-292.6 (-12)
35	-15.44 (-03)	27.64 (-06)	58.07 (-09)	-225.0 (-12)	-292.4 (-12)	-209.4 (-12)
36	-15.02 (-03)	25.19 (-06)	49.77 (-09)	-315.3 (-12)	-219.6 (-12)	-406.9 (-12)
37	-14.61 (-03)	23.02 (-06)	42.04 (-09)	980.6 (-12)	-91.51 (-12)	-172.3 (-12)
38	-14.23 (-03)	31.10 (-06)	37.07 (-09)	-2.417 (-09)	-262. (-12)	-179.4 (-12)
39	-13.86 (-03)	19.38 (-06)	32.15 (-09)	1.817 (-09)	-129.5 (-12)	-190.1 (-12)
40	-13.52 (-03)	17.84 (-06)	2A.04 (-09)	-1.033 (-09)	-244.6 (-12)	-125.4 (-12)
41	-13.19 (-03)	16.47 (-06)	34.52 (-09)	-123.0 (-12)	1.318 (-09)	536.4 (-12)
42	-12.88 (-03)	15.23 (-06)	21.52 (-09)	405.5 (-12)	38.49 (-12)	-14.98 (-12)
43	-12.58 (-03)	14.11 (-06)	18.95 (-09)	-412.9 (-12)	-7.890 (-12)	61.82 (-12)
44	-12.29 (-03)	13.10 t-06)	16.74 (-09)	212.2 (-12)	-185.9 (-12)	-78.36 (-12)
45	-12.02 (-03)	12.18 (-06)	14.83 (-09)	-574.3 (-12)	25.56 (-12)	-65.12 (-12)
46	-11.74 (-03)	11.35 (-06)	13.17 (-09)	1.011 (-09)	-164.2 (-12)	
47	-11.51 (-03)	10.59 (-06)	11.75 (-09)	-1.917 (-09)		
48	-11.27 (-03)	9.901 (-06)	10.49 (-09)			
49	-11.04 (-03)	9.268 (-06)				
50	-10.82 (-03)					

s-formula

n	a_n^0	a_n^1	a_n^2	a_n^3	a_n^4	a_n^5
1	-135.3 (-03)					
2	-135.3 (-03)	-135.3 (-03)				
3	-135.3 (-03)	-135.3 (-03)	-135.3 (-03)			
4	-135.3 (-03)	-135.3 (-03)	-135.3 (-03)	-135.3 (-03)		
5	- 52.00 (-03)	- 99.62 (-03)	786.8 (-03)	271.6 (-03)	54.41 (-03)	
6	- 85.34 (-03)	- 65.73 (-03)	- 16.49 (-03)	- 85.00 (-03)	-262.3 (-03)	-157.5 (-03)
7	- 71.45 (-03)	- 48.05 (-03)	- 69.79 (-03)	-193.1 (-03)	-123.0 (-03)	-422.0 (-03)
8	- 65.10 (-03)	-258.4 (-03)	-104.9 (-03)	- 72.73 (-03)	144.6 (-03)	40.11 (-03)
9	- 58.30 (-03)	- 4.047(-03)	8.465(-03)	977.3 (-06)	1.062(-03)	946.9 (-06)
10	- 52.87 (-03)	5.003(-03)	1.862(-03)	1.051(-03)	958.4 (-06)	1.069(-03)
11	- 48.29 (-03)	1.818(-03)	1.249(-03)	1.879(-03)	3.699(-03)	2.498(-03)
12	- 44.42 (-03)	1.442(-03)	4.906(-03)	2.423(-03)	1.913(-03)	36.16 (-03)
13	- 41.11 (-03)	1.002(-03)	141.6 (-06)	- 59.10 (-06)	16.82 (-06)	1.991(-06)
14	- 38.25 (-03)	740.1 (-06)	- 19.97 (-06)	9.156(-06)	3.567(-06)	2.219(-06)
15	- 35.76 (-03)	560.8 (-06)	8.591(-06)	3.888(-06)	2.455(-06)	3.161(-06)
16	- 33.56 (-03)	435.7 (-06)	4.049(-06)	2.876(-06)	3.935(-06)	622.4 (-09)
17	- 31.62 (-03)	345.5 (-06)	3.244(-06)	13.87 (-06)	6.874(-06)	4.572(-06)
18	- 29.89 (-03)	278.7 (-06)	2.359(-06)	52.90 (-09)	-151.5 (-09)	- 53.77 (-09)
19	- 28.34 (-03)	228.2 (-06)	1.753(-06)	-123.6 (-09)	- 62.43 (-09)	39.01 (-09)
20	- 26.94 (-03)	189.3 (-06)	1.317(-06)	- 46.75 (-09)	- 17.35 (-09)	29.42 (-09)
21	- 25.67 (-03)	158.8 (-06)	1.001(-06)	- 27.47 (-09)	4.419(-09)	-399.3 (-12)
22	- 24.51 (-03)	134.5 (-06)	770.6 (-09)	- 16.07 (-09)	-996.3 (-12)	-306.6 (-12)
23	- 23.45 (-03)	115.0 (-06)	600.0 (-09)	- 9.955(-09)	-295.2 (-12)	-174.0 (-12)
24	- 22.48 (-03)	99.09 (-06)	472.3 (-09)	- 6.407(-09)	-200.8 (-12)	545.3 (-12)
25	- 21.59 (-03)	85.99 (-06)	375.5 (-09)	- 4.261(-09)	-118.1 (-12)	- 33.54 (-12)
26	- 20.76 (-03)	75.12 (-06)	301.4 (-09)	- 2.913(-09)	- 78.46 (-12)	- 1.891(-09)
27	- 20.00 (-03)	66.01 (-06)	244.1 (-09)	- 2.040(-09)	- 37.84 (-12)	- 46.11 (-12)
28	- 19.29 (-03)	58.32 (-06)	199.3 (-09)	- 1.457(-09)	- 46.95 (-12)	- 38.25 (-12)
29	- 18.63 (-03)	51.78 (-06)	163.9 (-09)	- 1.061(-09)	8.653(-12)	- 21.50 (-12)
30	- 18.01 (-03)	46.19 (-06)	135.8 (-09)	-782.9 (-12)	- 44.37 (-12)	- 12.13 (-12)
31	- 17.43 (-03)	41.38 (-06)	113.3 (-09)	-588.5 (-12)	19.73 (-12)	- 8.873(-12)
32	- 16.89 (-03)	37.21 (-06)	95.04 (-09)	-446.0 (-12)	- 23.29 (-12)	- 6.089(-12)
33	- 16.38 (-03)	33.59 (-06)	80.22 (-09)	-343.2 (-12)	893.9 (-15)	- 2.267(-12)
34	- 15.90 (-03)	30.42 (-06)	68.08 (-09)	-266.6 (-12)	- 2.330(-12)	899.2 (-15)
35	- 15.44 (-03)	27.64 (-06)	58.07 (-09)	-209.0 (-12)	27.17 (-12)	2.405(-12)
36	- 15.02 (-03)	25.19 (-06)	49.77 (-09)	-164.0 (-12)	- 63.99 (-12)	39.06 (-12)
37	- 14.61 (-03)	23.02 (-06)	42.85 (-09)	-134.2 (-12)	- 5.393(-09)	- 2.425(-09)
38	- 14.23 (-03)	21.10 (-06)	37.05 (-09)	-104.3 (-12)	- 36.55 (-12)	- 43.72 (-12)
39	- 13.86 (-03)	19.38 (-06)	32.17 (-09)	- 84.44 (-12)	- 43.01 (-12)	- 35.52 (-12)
40	- 13.52 (-03)	17.84 (-06)	28.03 (-09)	- 71.60 (-12)	-197.6 (-12)	-109.9 (-12)
41	- 13.19 (-03)	16.47 (-06)	24.52 (-09)	- 57.06 (-12)	- 36.25 (-12)	- 66.55 (-12)
42	- 12.88 (-03)	15.23 (-06)	21.52 (-09)	- 48.98 (-12)	- 69.07 (-12)	- 51.27 (-12)
43	- 12.58 (-03)	14.11 (-06)	18.95 (-09)	- 34.01 (-12)	- 37.77 (-12)	- 33.69 (-12)
44	- 12.29 (-03)	13.10 (-06)	16.74 (-09)	- 38.23 (-12)	- 34.49 (-12)	- 40.05 (-12)
45	- 12.02 (-03)	12.18 (-06)	14.83 (-09)	- 23.58 (-12)	- 24.92 (-12)	- 23.26 (-12)
46	- 11.76 (-03)	11.35 (-06)	13.18 (-09)	- 24.85 (-12)	- 23.63 (-12)	
47	- 11.51 (-03)	10.59 (-06)	11.74 (-09)	- 17.96 (-12)		
48	- 11.27 (-03)	9.901(-06)	10.49 (-09)			
49	- 11.04 (-03)	9.268(-06)				
50	- 10.82 (-03)					

a-formula.

n	k_n^0	k_n^1	k_n^2	k_n^3	k_n^4	k_n^5
2	10.00(+39)					
3	- 2.000(+00)	- 2.000(+00)				
4	400.0 (-03)	-949.0 (-03)	071.1 (-03)			
5	-120.0 (+00)	- 1.057(+00)	863.S (-03)	108.7 (-03)		
6	-119.9 (-03)	- 1.497(+00)	-461.4 (-03)	-2.040(+00)	- 1.598(+00)	
7	- 1.062(+00)	1.666(+00)	1.372(+00)	-525.3 (-03)	-384.9 (-03)	792.6 i-03)
8	-948.0 t-o 3)	711.6 (-03)	411.4 (-03)	- 1.633(+00)	1.893(+00)	12.33(+00)
9	-249.6 (-03)	- 4.364 (+00)	5.356(+00)	- 1.910(+00)	- 2.037(+00)	- 1.917(+00)
10	5.585(+00)	1.536(+00)	- 2.043(+00)	-669.2 (-03)	7s7.3 (-03)	19.27(+00)
11	1.215(+00)	- 1.142(+00)	2.439(+00)	-1.3641+00)	- 1.059(+00)	899 [illegible] (-03)
12	1.307(+00)	-880.6 (-03)	664.3 (-03)	260.5 (-03)	-942.1 (-03)	931.o (-03)
13	1.201(+00)	401.6 (-03)	- 6.397(+00)	- 23.28(+00)	10.83(+00)	- 1.758(+00)
14	1.155(+00)	6.662(+00)	76.03(+00)	2.229(+00)	- 2.509(+00)	69S.6 (-03)
15	1.121(+00)	2.607(+00)	1.837(+00)	- 1.867(+00)	- 1.693(+00)	7.290(+00)
16	1.097(+00)	2.896(+00)	- 1.089(+00)	1.787(+00)	-236.5 (-03)	13.43(+00)
17	1.080(+00)	2.847(+00)	-924.1 (-03)	754.9 (-03)	443.9 (-03)	923.9 (-03)
18	1.067(+00)	2.861(+00)	1.442(+00)	- 4.161(+00)	- 3s.79(-03)	- 14.68 (+00)
19	1.057(+00)	2.870(+00)	49.21(+00)	567.9 (-03)	- 12.19(+00)	- 1.730(+00)
20	1.049(+00)	2.880(+00)	7.639(+00)	-102.3 (-03)	- 1.879(+00)	-307.3 (-03)
21	1.043(+00)	2.889(+00)	6.970(+00)	-4.344(+00)	390.9 (+00)	- 2.964(+00)
22	1.038(+00)	2.898(+00)	6.305(+00)	2.095(+00)	- 3.960(+00)	10.05 (+00)
23	1.033(+00)	2.906(+00)	5.994(+00)	- 4.819(+00)	- 6.165(+00)	-1.2401+00)
24	1.030(+00)	2.913(+00)	5.687(+00)	-511.7 (-03)	3.206(+00)	-674.5 (-03)
25	1.027(+00)	2.920(+00)	5.686(+00)	- 1.430(+00)	- 4.294(+00)	- 3.034(+00)
26	1.024(+00)	2.926(+00)	5.411(+00)	-950.1 (-03)	2.307(+00)	-242.7 (-03)
27	1.022(+00)	2.931(+00)	5.509(+00)	- 1.049(+00)	649.8 (-03)	73s.7 (-03)
28	1.020(+00)	2.936(+00)	5.248(+00)	- 1.200(+00)	- 2.467(+00)	-529.4 (-03)
29	1.018(+00)	2.941 (+00)	5.415(+00)	-813.3 (-03)	1.031(+00)	- 1.476(+00)
30	1.017(+00)	2.945(+00)	5.255(+00)	- 2.335(+00)	491.9 (-03)	- 3.625(+00)
31	1.015(+00)	2.948(+00)	5.065(+00)	-1.7131+00)	- 1.728(+00)	- 1.760(+00)
32	1.014(+00)	2.951(+00)	5.763(+00)	312.6 (-03)	-26.12(-03)	-420.9 (-03)
33	1.013(+00)	2.955(+00)	4.550(+00)	- 12.30(+00)	- 4.324(+00)	- 21.52 (+00)
34	1.012(+00)	2.957(+00)	5.547(+00)	8.058(+00)	- 2.101(+00)	- 2.181(+00)
35	1.012(+00)	2.960(+00)	4.693(+00)	- 2.705(+00)	- 1.739(+00)	2.243(+00)
36	1.011(+00)	2.962(+00)	92..67(+00)	3.728(+00)	-426.5 (-03)	565.8 (-03)
37	1.010(+00)	2.964(+00)	522.5 (-03)	4.914(+00)	6.448(+00)	- 1.662(+00)
38	1.009(+00)	2.966(+00)	- 2.813(+00)	5 [illegible])	-39.66(+00)	347.4 (-03)
39	1.009(+00)	2.968(+00)	344.2 (-03)	5.237(+00)	- 3.137(+00)	- 4.054(+00)
40	1.008(+00)	2.970(+00)	- 21.46(+00)	-385.8f-03)	1.078(+00)	495.3 (-03)
41	1.008(+00)	2.971(+00)	48.97 (+00)	- 1.502(+00)	1.050(+00)	2.010(+00)
42	1.008(+00)	2.973(+00)	1.312(+00)	4.728(+00)	- 1.719(+00)	- 2.910(+00)
43	1.007(+00)	2.974(+00)	- 15.55 (+00)	- 9.059(+00)	237.0 (-03)	434.0 (-03)
44	1.007(+00)	2.975(+00)	959.3 (-03)	- 9.983(+00)	13.09(+00)	
45	1.006(+00)	2.976(+00)	- 2.589(+00)	49.67(+00)		
46	1.006(+00)	2.978(+00)	-467.3 i-031			
47	1.006(+00)	2.978(+00)				
48	1.006(+00)					

k-estimates

n	ρ_n^0	ρ_n^2	ρ_n^4	ρ_n^6	ρ_n^8	ρ_n^{10}
1	-135.3 (-03)	-135.3 (-03)	86.89 (-03)	- 25.69 (-03)	- 25.97 (-03)	37.75 (-03)
2	-135.3 (-03)	-135.3 (-03)	-107.0 (-03)	- 44.27 (-03)	- 1.065(-03)	2.425(-03)
3	-135.3 (-03)	31.33 (-03)	- 42.00 (-03)	- 41.68 (-03)	20.25 (-03)	1.268(-03)
4	-135.3 (-03)	- 99.62 (-03)	- 43.90 (-03)	- 5.451(-03)	251.9 (-06)	479.6 (-06)
5	- 52.00 (-03)	- 65.73 (-03)	- 88.66 (-03)	10.74 (-03)	1.138(-03)	152.7 (-06)
6	- 85.34 (-03)	- 48.05 (-03)	- 3.792(-03)	1.323(-03)	1.373(-03)	-134.3 (-06)
7	- 71.45 (-03)	-258.4 (-03)	7.172(-03)	1.160(-03)	166.4 (-06)	19.78 (-06)
8	- 65.10 (-03)	- 4.047(-03)	1.774(-03)	2.790(-03)	- 72.54 (-06)	- 1.444(-06)
9	- 58.30 (-03)	5.003(-03)	1.246(-03)	137.7 (-06)	10.26 (-06)	1.742(-06)
10	- 52.87 (-03)	1.818(-03)	4.780(-03)	- 39.97 (-06)	314.6 (-09)	946.3 (-09)
11	- 46.29 (-03)	1.442(-03)	132.8 (-06)	5.304(-06)	1.483(-06)	183.2 (-09)
12	- 44.42 (-03)	1.002(-03)	- 27.45 (-06)	1.015(-06)	1.162(-06)	-370.0 (-09)
13	- 41.11 (-03)	740.1 (-06)	4.360(-06)	1.420(-06)	- 66.08 (-09)	-155.3 (-09)
14	- 38.25 (-03)	560.8 (-06)	1.291(-06)	1.304(-06)	-327.7 (-09)	- 18.51 (-09)
15	- 35.76 (-03)	435.7 (-06)	1.403(-06)	- 35.08 (-09)	-151.0 (-09)	4.006(-09)
16	- 33.56 (-03)	345.5 (-06)	1.088(-06)	-252.2 (-09)	- 21.97 (-09)	-704.7 (-12)
17	- 31.62 (-03)	278.7 (-06)	852.3 (-09)	- 90.36 (-09)	4.678(-09)	- 64.60 (-12)
18	- 29.89 (-03)	228.2 (-06)	663.2 (-09)	- 46.51 (-09)	-690.8 (-12)	-291.8 (-12)
19	- 28.34 (-03)	189.3 (-06)	517.8 (-09)	- 25.07 (-09)	- 46.32 (-12)	-128.4 (-12)
20	- 26.94 (-03)	158.8 (-06)	406.6 (-09)	- 14.60 (-09)	-375.8 (-12)	-164.2 (-12)
21	- 25.67 (-03)	134.5 (-06)	321.7 (-09)	- 8.937(-09)	363.6 (-12)	42.37 (-12)
22	- 24.51 (-03)	115.0 (-06)	256.5 (-09)	- 5.745(-09)	-749.5 (-12)	- 95.29 (-12)
23	- 23.45 (-03)	99.09 (-06)	206.1 (-09)	- 3.763(-09)	4.018(-09)	98.19 (-12)
24	- 22.48 (-03)	85.99 (-06)	166.9 (-09)	- 2.638(-09)	- 1.138(-09)	-147.0 (-12)
25	- 21.59 (-03)	75.12 (-06)	136.1 (-09)	- 1.700(-09)	- 2.277(-09)	-104.9 (-12)
26	- 20.76 (-03)	66.01 (-06)	111.8 (-09)	- 1.378(-09)	-696.1 (-12)	20.14 (-09)
27	- 20.00 (-03)	58.32 (-06)	92.50 (-09)	-836.9 (-12)	-959.2 (-12)	-701.3 (-12)
28	- 19.29 (-03)	51.78 (-06)	76.99 (-09)	-736.6 (-12)	3.108(-09)	- 22.42 (-12)
29	- 18.63 (-03)	46.19 (-06)	64.47 (-09)	-530.5 (-12)	-339.0 (-12)	-299.9 (-12)
30	- 18.01 (-03)	41.38 (-06)	54.30 (-09)	-337.5 (-12)	-337.4 (-12)	-344.6 (-12)
31	- 17.43 (-03)	37.21 (-06)	45.97 (-09)	-338.8 (-12)	-306.9 (-12)	- 90.74 (-12)
32	- 16.89 (-03)	33.59 (-06)	39.12 (-09)	-149.1 (-12)	-195.9 (-12)	-159.3 (-12)
33	- 16.38 (-03)	30.42 (-06)	33.45 (-09)	-663.5 (-12)	-139.3 (-12)	-141.8 (-12)
34	- 15.90 (-03)	27.64 (-06)	28.74 (-09)	1.226(-09)	-166.2 (-12)	- 15.14 (-12)
35	- 15.44 (-03)	25.19 (-06)	24.78 (-09)	- 2.458(-09)	- 39.69 (-12)	- 40.70 (-12)
36	- 15.02 (-03)	23.02 (-06)	21.49 (-09)	1.344(-09)	- 21.11 (-12)	- 85.69 (-12)
37	- 14.61 (-03)	21.10 (-06)	18.67 (-09)	-603.6 (-12)	-247.2 (-12)	61.63 (-12)
38	- 14.23 (-03)	19.38 (-06)	16.30 (-09)	63.40 (-12)	- 59.17 (-12)	- 41.38 (-12)
39	- 13.86 (-03)	17.84 (-06)	14.28 (-09)	-408.4 (-12)	24.37 (-12)	- 45.73 (-12)
40	- 13.52 (-03)	16.47 (-06)	12.55 (-09)	411.8 (-12)	-355.2 (-12)	105.6 (-12)
41	- 13.19 (-03)	15.23 (-06)	11.06 (-09)	-194.3 (-12)	-200.9 (-12)	
42	- 12.98 (-03)	14.11 (-06)	9.783(-09)	-309.7 (-12)	315.6 (-12)	
43	- 12.58 (-03)	13.10 (-06)	8.676(-09)	207.6 (-12)		
44	- 12.29 (-03)	12.18 (-06)	7.712(-09)	286.8 (-12)		
45	- 12.02 (-03)	11.35 (-06)	6.877(-09)			
46	- 11.76 (-03)	10.59 (-06)	6.152(-09)			
47	- 11.51 (-03)	9.901(-06)				
48	- 11.27 (-03)	9.268(-06)				
49	- 11.04 (-03)					
50	- 10.82 (-03)					

ρ - formula

n	r_n^0	r_n^2	r_n^4	r_n^6	r_n^8	r_n^{10}
1	-135.3 (-03)	-135.3 (-03)	-135.3 (-03)	-135.3 (-03)	-151.o (-03)	156.6 (-03)
2	-135.3 (-03)	-1.35.3 (-03)	-135.3 (-03)	-112.2 (-03)	-39.33 (-03)	-5.771 (-03)
3	-135.3 (-03)	-135.3 (-03)	-206.3 (-03)	Ii13.3 (-03)	8.615 (-03)	1.207 (-03)
4	-135.3 (-03)	-99.62 (-03)	-43.50 (-03)	-5.451 (-03)	251.9 (-06J	479.6 (-06)
5	-52.00 (-03)	-65.73 (-03)	-88.66 (-03)	10.74 (-03)	1.138 (-03)	152.7 (-06)
6	-es.34 (-03)	-4U.05 (-03)	-3.792 (-03)	1.323 (-03)	1.373 (-03)	-134.3 (-06)
7	-71.45 (-03)	-258.4 (-03)	7.172 (-03)	1.160 (-03)	166.4 (-06)	19.78 (-06)
8	-65.10 (-03)	-4.047 (-03)	1.774 (-03)	2.790 (-03)	-72.54 (-06)	-1.444 (-06)
9	-59.30 (-03)	5.003 (-03)	1.246 (-03)	137.7 (-06J	10.26 (-06)	1.742 (-06)
10	-52.877 (-03)	1.818 (-33)	4.780 (-03)	-39.97 (-06)	314.6 (-09)	946.3 (-09)
11	-48.29 (-03)	1.442 (-03)	132.8 (-06)	5.304 (-06)	1.483 (-06)	183.2 (-09)
12	-44.4% (-03)	1.002 (-03)	-27.45 (-06)	1.015 (-06)	1.162 (-06)	-370.0 (-09)
13	-41.11 (-03)	740.1 (-06)	4.360 (-06)	1.420 (-06)	-66.07 (-09)	-155.3 (-09)
14	-38.25 (-03)	560.8 (-06)	1 ● 291 (-06)	1.304 (-06)	-327.7 (-09J	-18.52 (-09)
15	-35.76 (-03)	435.7 (-06)	1.403 (-06)	-35.08 (-09)	-151.0 (-09)	4.048 (-09)
16	-33.56 (-03)	345.5 (-06)	1.088 (-06)	-292.2 (-09)	-21.98 (-09)	-70.08.8 (-12)
17	-31.62 (-03)	278.7 (-06)	852.3 (-09)	-90.36 (-09)	4.730 (-09)	-43.65 (-12)
18	-25.89 (-03)	228.2 (-06)	6G3.2 (-09)	-46.51 (-09)	-734.2 (-12)	-281.9 (-12)
19	-20.34 (-03)	189.3 (-06)	517.e (-09)	-25.06 (-09)	-14.52 (-12)	-118.8 (-12)
20	-26.94 (-03)	158.8 (-06)	406.6 (-09)	-14.60 (-09)	-368.3 (-12)	-171.7 (-12)
21	-25.67 (-C3)	134.5 (-06)	321.7 (-09)	-8.936 (-09)	288.4 (-12)	40.53 (-12)
22	-24.51 (-03)	115.3 (-06)	256.5 (-09)	-5.742 (-09)	-673.4 (-12)	-87.12 (-12)
23	-23.45 (-03)	39.09 (-06)	206.1 (-09)	-3.768 (-09)	3.686 (-09)	82.06 (-12)
24	-22.48 (-C3)	85.95 (-06)	166.9 (-09)	-2.633 (-09)	-1.174 (-09)	-113.5 (-12)
25	-21.59 (-03)	75.12 (-06)	136.1 (-09)	-1.697 (-09)	-2.153 (-09)	-197.4 (-12)
26	-23.76 (-03)	66.01 (-06)	111.0 (-09)	-1.390 (-09)	-700.9 (-12)	-1.629 (-09)
27	-'0.00 (-03)	58.32 (-06)	92.50 (-09)	-326.1 (-12)	-938 ● 7 (-12)	-843.1 (-12)
28	-19.29 (-03)	51.78 (-06)	76.59 (-09)	-734.5 (-12)	-1.622 (-09)	-40.03 (-12)
29	-18.63 (-03)	46.19 (-06)	64.47 (-09)	-546.9 (-12)	-355.9 (-12)	-316.1 (-12)
30	-Id.01 (-03)	41.38 (-C6)	54.3C (-09)	-313.4 (-12)	-316.2 (-12)	-316.3 (-12)
31	-17.43 (-03)	37.21 (-06)	45.97 (-09)	-357.6 (-12)	-316.1 (-12)	-82.05 t-12)
32	-16.89 (-03)	33.55 (-06)	39.12 (-09)	-158.8 (-12)	-195.4 (-12)	-159.1 (-12)
33	-16.38 (-03)	30.4% (-06)	33.4s (-09)	-611.1 (-12)	-132 ● 7 (-12)	-139.0 (-12)
34	-15.90 (-03)	27.64 (-06)	213.74 (-09)	1.151 (-09)	-172.1 (-12)	67.30 (-12)
35	-15.44 (-C3)	25.19 (-06)	24.79 (-09)	-2.339 (-09)	-44.04 (-12)	-43.51 (-12)
36	-15.02 (-03)	23.02 (-06)	21.49 (-09)	1.273 (-09)	10.12 (-12)	-82.09 (-12)
37	-14.61 (-C3)	21.10 (-06)	18.67 (-09)	-573.5 (-12)	-254.5 (-12)	65.49 (-12)
38	-14.23 (-03)	19.38 (-06)	16.30 (-09)	1C7.9 (-12)	-59.11 (-12)	-42.97 (-12)
39	-13.86 (-03)	17.84 (-06)	14.29 (-09)	-507.4 (-12)	23.87 (-12)	-43.90 (-12)
40	-13.52 (-03)	16.47 (-06)	12.55 (-09)	502.9 (-12)	-344.2 (-12)	71.29 (-12)
41	-13.19 (-03)	15.23 (-06)	11.06 (-09)	-257.7 (-12)	-255.4 (-12)	
42	-12.88 (-C3)	14.11 (-06)	9.783 (-09)	-326.9 (-12)	191.2 (-12)	
43	-12.58 (-03)	13.10 (-06)	8.676 (-09)	288.7 (-12)		
44	-12.29 (-03)	12.18 (-06)	7.712 (-09)	188.6 (-12)		
45	-12.02 (-03)	11.35 (-06)	6.877 (-09)			
46	-11.76 (-03)	10.59 (-06)	6.152 (-09)			
47	-11.51 (-C3)	9.901 (-06)				
48	-IA.27 (-03)	9.268 (-06)				
49	-11.04 (-03)					
50	-10.82 (-03)					

rational extrapolation

n	p_n^0	p_n^2	p_n^4	p_n^6	p_n^8	p_n^{S0}
1	-135.3 (-03)	-135.3 (-03)	2.035(+00)	17.99 (+00)	43.94 (+00)	67.84(+00)
2	-135.3 (-03)	-135.3 (-03)	-6.116(+00)	-24.77 (+00)	-46.16 (+00)	-66.83 (+00)
3	-135.3 (-03)	906.3 (-03)	8.477(+00)	22.21 (+00)	37.89 (+00)	54.23 (+00)
4	-135.3 (-03)	-1.319(+00)	-6.195(+00)	-14.20 (+00)	-25.23 (+00)	-36.22 (+00)
5	-52.00 (-03)	671.6 (-03)	2.694(+00)	7.197(+00)	13.73 (+00)	19.89(+00)
6	-85.34 (-03)	-118.3 (-03)	-952.4 (-03)	-3.035(+00)	-6.027(+00)	-9.048(+00)
7	-71.45 (-03)	54.57 (-03)	314.8 (-03)	1.006(+00)	2.143(+00)	3.468(+00)
8	-65.10 (-03)	-4.047(-03)	-66.26 (-03)	-261.2 (-03)	-642.1 (-03)	-1.133(+00)
9	-58.30 (-03)	3.835(-03)	13.62 (-03)	61.29 (-03)	165.5 (-03)	314.9 (-03)
10	-52.87 (-03)	1.497(-03)	-2.361(-03)	-11.96 (-03)	-35.58 (-03)	-75.01 (-03)
11	-48.29 (-03)	1.237(-03)	373.7 (-06)	1.891(-03)	6.677(-03)	15.73 (-03)
12	-44.42 (-03)	884.2 (-06)	-3.063(-06)	-311.5 (-06)	-1.139(-03)	-2.894(-03)
13	41.11 (-03)	667.[illegible] (-06)	35.76 (-06)	34.49 (-06)	158.9 (-06)	464.3 (-06)
14	-38.25 (-03)	513.7 (-06)	20.86 (-06)	-5.247(-06)	-22.55 (-06)	-68.93 (-06)
15	-35.76 (-03)	404.1 (-06)	15.01 (-06)	374.7 (-09)	2.295(-06)	8.803(-06)
16	-33.56 (-03)	323.6 (-06)	10.63 (-06)	-1.382(-09)	-379.9 (-09)	-1.082(-06)
17	-31.62 (-03)	263.2 (-06)	7.697(-06)	[illegible] (-09)	-20.38 (-09)	107.1 (-09)
18	-29.89 (-03)	216.9 (-06)	5.668(-06)	73.79 (-09)	-25.60 (-09)	-26.94 (-09)
19	-28.34 (-03)	180.9 (-06)	4.243(-06)	61.69 (-09)	-11.30 (-09)	24.69 (-09)
20	-26.94 (-03)	152.5 (-06)	3.224(-06)	48.65 (-09)	-7.177(-09)	-50.05 (-09)
21	-25.67 (-03)	129.7 (-00)	2.484(-06)	37.59 (-09)	-2.000(-09)	98.49 (-09)
22	-24.51 (-03)	111.2 (-06)	1.938(-06)	28.75 (-09)	-4.503(-09)	-179.1 (-09)
23	-23.45 (-03)	96.14 (-06)	1.529(-06)	22.07 (-09)	3.993(-09)	256.6 (-09)
24	-22.48 (-03)	83.65 (-06)	1.219(-06)	16.82 (-09)	-8.081 (-09)	-296.1 (-09)
25	-21.59 (-03)	73.24 (-06)	981.9 (-09)	13.18 (-09)	7.928(-09)	257.7 (-09)
26	-20.76 (-03)	64.48 (-06)	797.8 (-09)	9.976(-09)	-7.768(-09)	-118.3 (-09)
27	-20.00 (-03)	57.07 (-06)	653.5 (-09)	8.030(-09)	3.726(-09)	-63.02 (-09)
28	-19.29 (-03)	50.75 (-06)	539.4 (-09)	6.126(-09)	1.246(-09)	206.2 (-09)
29	-18.63 (-03)	45.34 (-06)	[illegible] (-09)	4.883(-09)	-4.299(-09)	-565.4 (-09)
30	-18.01 (-03)	40.66 (-06)	375.2 (-09)	3.975(-09)	5.809(-09)	2.170(-06)
31	-17.43 (-03)	36.61 (-06)	315.9 (-09)	3.078(-09)	-22.40 (-09)	-6.176(-06)
32	-16.89 (-03)	33.08 (-06)	[illegible] (-09)	2.625(-09)	85.02 (-09)	11.50 (-06)
33	-16.38 (-03)	29.99 (-06)	227.7 (-09)	1.417(-09)	-188.8 (-09)	-14.22 (-06)
34	-15.90 (-03)	27.27 (-06)	194.8 (-09)	3.570(-09)	248.6 (-09)	11.92 (-06)
35	-15.44 (-03)	24.87 (-06)	167.5 (-09)	-1.443(-09)	-197.9 (-09)	-7.417(-06)
36	-15.02 (-03)	22.75 (-06)	144.6 (-09)	3.211(-09)	98.81 (-09)	5.257(-06)
37	-14.61 (-03)	20.86 (-06)	125.4 (-09)	202.5 (-12)	-55.73 (-09)	-5.881(-06)
38	-14.23 (-03)	19.17 (-06)	109.1 (-09)	1.041(-09)	75.57 (-09)	5.764(-06)
39	-13.86 (-03)	17.66 (-06)	95.36 (-09)	38.09 (-12)	-85.75 (-09)	-2.768(-06)
40	-13.52 (-03)	16.31 (-06)	83.62 (-09)	1.353(-09)	41.66 (-09)	-324.7 (-09)
41	-13.19 (-03)	15.09 (-06)	73.57(-09)	49.46 (-12)	17.07 (-09)	
42	-12.88 (-03)	13.99 (-06)	64.94 (-09)	155.8 (-12)	-10.97 (-09)	
43	-12.58 (-03)	12.99 (-06)	57.49 (-09)	624.6 (-12)		
44	-12.29 (-03)	12.09 (-06)	51.05 (-09)	695.5 (-12)		
45	-12.02 (-03)	11.27 (-06)	45.45 (-09)			
46	-11.76 (-03)	10.52 (-06)	40.58 (-09)			
47	-11.51 (-03)	9.833(-06)				
48	-11.27 (-03)	9.207(-06)				
49	-11.04 (-03)					
50	-10.82 (-03)					

polynomial extrapolation

n	t_n^0	t_n^2	t_n^4	t_n^6	t_n^8	t_n^{10}
1	-135.3 (-03)	-135.3 (-03)	1.308(+00)	6.366(+00)	7.419(+00)	5.856(+00)
2	-135.3 (-03)	-135.3 (-03)	- 3.630(+00)	- 7.464(+00)	- 6.670(+00)	- 4.927(+00)
3	-135.3 (-03)	817.0 (-03)	4.912(+00)	6.493(+00)	5.519(+00)	3.830(+00)
4	-135.3 (-03)	- 1.200(+00)	- 3.611(+00)	- 4.439(+00)	- 3.970(+00)	- 2.630(+00)
5	-52.00 (-03)	C11.6 (-03)	1.656(+00)	2.509(+00)	2.365(+00)	1.563(+00)
6	-05.34 (-03)	-110.8 (-03)	-631.5 (-03)	- 1.155(+00)	- 1.145(+00)	-799.0 (-03)
7	-71.45 (-03)	50.67(-03)	214.5 (-03)	410.5 (-03)	456.5 (-03)	349.8 (-03)
8	- 65.10 (-03)	- 4.042(-03)	-47.31(-03)	-116.2 (-03)	-123.8 (-03)	-130.1 (-03)
9	-59.30 (-03)	3.459(-03)	9.934(-03)	29.16 (-03)	43.63 (-03)	41.08 (-03)
10	-52.37 (-03)	1.337(-03)	- 1.833(-03)	- 5.994(-03)	-10.30(-03)	- 11.13 (-03)
11	-40.29 (-03)	1.123(-03)	254.4 (-06)	1.010(-03)	2.121(-03)	2.626(-03)
12	- 44.42 (-03)	806.2 (I-06)	- 24.83 (-06)	-172.3 (-06)	-386.2 (-06)	-535.9 (-06)
13	-41.11 (-03)	611.6 (-06)	13.70(-06)	19.74(-06)	58.64(-06)	95.69(-06)
14	-38.25 (-03)	473.0 (-06)	6.721(-06)	- 3.409(-06)	- 8.615(-06)	-15.44(-06)
15	- 35.76 (-03)	373.6 (-06)	5.177(-06)	13.10 (-09)	1.003(-06)	2.163(-06)
16	- 33.56 (-03)	300.3 (-06)	3.690(-06)	-163.5 (-09)	-136.6 (C-09)	-280.1 (-09)
17	-31.02 (-03)	24S.1 (-06)	2.689(-06)	- 64.65 (-09)	4.067(-09)	33.7s (-09)
18	-29.09 (-03)	202.6 (-06)	1.984(-06)	-35.80(-09)	- 5.349(-09)	- 9.559(-09)
19	-28.34 (-03)	165.5 (-06)	1.485(-06)	-19.49(-09)	- 1.836(-09)	11.98(-09)
20	- 26.94 (-03)	143.2 (-06)	1.126(-06)	-11.06(-09)	-1.821(-09)	-21.58 (-09)
21	-25.67 (-03)	122.1 (-06)	865.6 (-09)	- 6.410(-09)	151.4 (-12)	44.28 (-09)
22	-24.51 (-03)	105.0 (-06)	673.3 (-09)	- 3.851(-09)	- 1.967(-09)	-81.03(-09)
23	-23.45 (-03)	90.93(-06)	529.6 (-09)	- 2.293(-09)	2.873(-09)	117.4 (-09)
24	-22.40 (-03)	79.28 (-06)	420.9 (-09)	- 1.497(-09)	- 4.701(-09)	-136.8 (-09)
25	- 21.59 (-03)	69.53(-06)	337. e (-09)	-77ts.s (-12)	5.052(-09)	119.2 (-09)
26	-20.76 (-03)	61.33(-06)	273.5 (-09)	-664.4 (-12)	- 4.749(-09)	-51.62(-09)
27	- 20.00 (-03)	54.37(-06)	223.2 (-09)	-245.7 (-12)	2.324(-09)	-36.29 (-09)
28	-19.29 (-03)	48.42(-06)	183.6 (-09)	-279.4 (-12)	938.0 (-12)	107.3 (-09)
29	- 18.63 (-03)	43.31(-06)	152.1 (-09)	-155.1 (-12)	- 2.810(-09)	-305. 9(-09)
30	- 18.01 (-03)	38.90(-06)	126.9 (-09)	-27.41(-12)	3.094 (-09)	1.201(-06)
31	-17.43 (-03)	35.07(-06)	106.4 (-09)	- 88.39 (-12)	-15.45(-09)	- 3.430(-06)
32	-16.89 (-03)	31.73(-06)	89.85 (-09)	58.47 (-12)	59.00(-09)	6.378(-06)
33	- 16.38 (-03)	28.79(-06)	76.25(-09)	-527.0 (-12)	-130.9 (-09)	- 7.867(-06)
34	- 15.90 (-03)	26.21(-06)	65.04(-09)	1.563(-09)	172.2 (-09)	6.587(-06)
35	-15.44 (-03)	23.93(-06)	55.74(-09)	- 2.305(-09)	-136.8 (-09)	- 4.131(-06)
36	-15.02 (-03)	21.91(-06)	48.02(-09)	1.717(-09)	68.43(-09)	3.0027 (-06)
37	-14.61 (-03)	20.11(-06)	41.50(-09)	-5co.1 (-12)	-39.60 (-09)	- 3.464(-06)
38	-14.23 (-03)	18.50(-06)	36.C4(-09)	231.1 (-12)	54 .ss (-09)	3.382(-06)
39	- 13.86 (-03)	17.06(-06)	31.41(-09)	-5C3.8 (-12)	-E1.72 (-09)	- 1.574(-06)
40	-13.52 (-03)	15.76(-06)	27.48(-09)	t07.6 (-12)	29.45(-09)	-237.8 (-09)
41	-13.19 (-03)	14.59(-06)	24.11(-09)	-332.1 (-12)	13.08(-09)	
42	- 12.88 (-03)	13.54(-06)	21.24(-09)	-192.0 (-12)	- 7.637(-09)	
43	-12.58 (-03)	12.58(-06)	18.76(-09)	257.1 (-12)		
44	-12.29 (-03)	11.71 (-06)	16.62 (-09)	364.5 (-12)		
45	-12.02 (-03)	10.92(-06)	14.77(-09)			
46	-11.76 (-03)	10.20(-06)	13.16(-09)			
47	- 11.51 (-03)	5.546(-06)				
48	-11.27 (-03)	8.944(-06)				
49	-11.04 (-03)					
50	- 10.82 (-03)					

elimination - first method*

n	q_n^0	q_n^2	q_n^4	q_n^6	q_n^8	q_n^{10}
1	-135.3 (-03)					
2	-135.3 (-03)	[illegible] (-03)				
3	-135.3 (-03)	-135.3 [illegible]	2.289(+00)			
4	-135.3 (-03)	905.3 (-03)	-5.697(+00)	24.36 (+00)		
5	-52.00 (-03)	-1.319(+00)	9.108(+00)	-31.71 (+00)	74.72 (+00)	
6	-85.34 (-03)	671.6 (-03)	-6.564(+00)	27.18 (+00)	-71.50 (+00)	144.0 (+00)
7	-71.45 (-03)	-119.3 (-03)	2.817(+00)	-16.69 (+00)	54.15 (+00)	-125.0 (+00)
8	-65.10 (-03)	54.57 (-03)	-983.8 (-03)	8.165(+00)	-33.74 (+00)	91.64 (+00)
9	-58.30 (-03)	-4.047 (-03)	323.4 (-03)	-3.359(+00)	17.46(+00)	-56.52 (+00)
10	-52.87 (-03)	1.835(-03)	-67.66 (-03)	1.095(+00)	-7.378(+00)	29.1[illegible] (+00)
11	-48.29 (-03)	1.497(-03)	13.87 (-03)	-279.9 (-03)	2.542(+00)	-12.55 (+00)
12	-44.42 (-03)	1.237(-03)	-2.390(-03)	64.95 (-03)	-741.2 (-03)	4.592(+00)
13	-41.11 (-03)	884.2 [illegible]	382.1 (-06)	-12.57 (-03)	187.2 (-03)	-1.444(+00)
14	-38.25 (-03)	667.2 [illegible]	-701.5 [illegible]	1.972(-03)	-39.57 (-03)	389.1 (-03)
15	-35.76 (-03)	513.7 (-06)	37.60 (-06)	-322.9 (-06)	7.317(-03)	-90.22 (-03)
16	-33.56 (-03)	[illegible] (-06)	22.01 (-06)	35.82 [illegible]	-1.235(-03)	[illegible]3.49 (-03)
17	-31.62 (-03)	323.6 [illegible]	15.78 (-06)	-5.256(-06)	170.5 (-06)	[illegible] (-03)
18	-29.89 (-03)	263.[illegible] (-06)	11.15 (-06)	473.7 (-09)	-24.05 [illegible]	526.9 (-06)
19	-28.34 (-03)	216.9 (-06)	8.061(-06)	56.06 (-09)	2.416(-06)	-77.25 (-06)
20	-26.54 (-03)	180.9 [illegible]	5.928(-06)	120.8 (-09)	-402.7 (-09)	9.732(-06)
21	-25.67 (-03)	152.5 (-06)	4.432(-06)	[illegible] (-09)	-22.49 (-09)	-1.186(-06)
22	-24.51 (-03)	129.7 (-06)	3.364(-06)	79.50 (-09)	-26.56 (-09)	106.5 (-09)
23	-23.45 (-03)	111.2 (-06)	2.589(-06)	60.95 (-09)	-12.18 (-09)	-14.01 (-09)
24	-22.48 (-03)	96.14 (-06)	2.018(-06)	46.18 (-09)	-6.361(-09)	-575.3 (-12)
25	-21.59 (-03)	83.65 (-06)	[illegible].592(-06)	34.89 (-09)	-3.228(-09)	-15.04 (-09)
26	-20.76 (-03)	[illegible].24 (-06)	[illegible].268(-06)	26.44 (-09)	-2.978(-09)	60.73 (-09)
27	-20.00 (-03)	64.48 (-06)	1.021(-06)	20.05 (-09)	2.993(-09)	-160.8 (-09)
28	-19.29 (-03)	57.0[illegible] (-06)	[illegible] (-09)	15.55(-09)	-8.305(-09)	[illegible] (-09)
29	-13.63 (-03)	50.75 (-06)	678.6 (-09)	11.72(-09)	9.613(-09)	-360.0 (-09)
30	-18.01 (-03)	45.34 (-06)	559.8 (-09)	9.400(-09)	-9.739(-09)	351.4 (-09)
31	-17.43 (-03)	40.66 (-06)	463.[illegible] (-09)	7.123(-09)	6.484(-09)	-25[illegible] (-09)
32	-16.89 (-03)	36.61 (-06)	389.0 (-09)	5.735(-09)	-[illegible]122(-09)	105.2 (-09)
33	-16.38 (-03)	33.[illegible] (-06)	327.4 (-09)	4.514(-09)	-1.235(-09)	194.3 (-09)
34	-15.90 (-03)	29.[illegible] (-06)	277.[illegible] (-09)	3.529(-09)	11.84 (-09)	-1.081(-06)
35	-15.44 [illegible]	27.27 (-06)	235.8 (-09)	3.233(-09)	-44.86 (-09)	3.564(-06)
36	-15.02 (-03)	24.87 (-06)	201.7 (-09)	1.288(-09)	123.9 (-09)	-8.429(-06)
37	-14.61 (-03)	22.75 (-06)	173.3 (-09)	4.304(-09)	-232.[illegible] (-09)	14.08 (-06)
38	-14.23 (-03)	20.86 (-06)	149.6 (-09)	-1.579(-09)	283.2 (-09)	-16.54 (-06)
39	-13.86 (-03)	19.17 [illegible]	129.6 (-09)	3.546(-09)	-223.5 (-09)	13.93 (-06)
40	-13.52 (-03)	17.66 (-06)	112.8 (-09)	109.9 (-12)	125.6 (-09)	-9.443(-06)
41	-13.19 (-03)	16.31 (-06)	98.56 (-09)	1.416(-09)	-84.45 (-09)	6.966(-06)
42	-12.88 (-03)	15.09 (-06)	86.41 [illegible]	-61.94 (-12)	84.10 (-09)	-5.494(-06)
43	-12.58 (-03)	13.99 (-06)	76.00 (-09)	1.145(-09)	-36.73 (-09)	1.825(-06)
44	-12.29 (-03)	12.99 (-06)	67.06 (-09)	1.021(-09)	-40.42 (-09)	797.6 (-09)
45	-12.02 (-03)	12.09 (-06)	59.37 (-09)	-68.15 (-12)	-11.57 (-09)	6.371(-06)
46	-11.76 (-03)	11.27 (-06)	52.70 (-09)	-1.098(-09)	235.1 (-09)	
47	-11.51 (-03)	10.52 (-06)	46.90 (-09)	3.444(-09)		
48	-11.27 (-03)	9.833(-06)	41.88 (-09)			
49	-11.04 (-03)	9.207(-06)				
50	-10.82 (-03)					

elimination - second method

As it happens, the coefficients c_j in the asymptotic expansion may be computed for this problem.

$$p_n = p_\infty\left(1 - 4n^{-1} + \frac{20}{3} n^{-3} + \frac{58}{3} n^{-4} + \frac{736}{15} n^{-5} + \frac{\;}{45} n^{-6} + \frac{\;}{63} n^{-7} + O(n^{-8})\right) \tag{6.3}$$

However, the amount of labor required for this computation makes the extrapolation schemes quite attractive.

Example 2.

To illustrate that k need not be an integer, on the following pages we give the corresponding results for

$$\zeta(1.5) = 2.6123753\,48685488$$

$$s_n^0 = \sum_{k=1}^{n} k^{-1.5}$$

By the Euler - Maclaurin expansion

$$\zeta(1.5) = s_n + n^{-.5}\left(2 - \frac{1}{2} n^{-1} + \frac{1}{8} n^{-2} + \cdots\right) \quad . \tag{6.4}$$

(Of course with the knowledge of the precise expansion (6.4) one can compute $\zeta(1.5)$ more efficiently.) A more serious application with fractional k occurs in contour integration (Lyness and Delves (1967)).

Finally, we note that the s-formula has been used to compute the asymptotic solution of a system of ordinary differential equations in quantum chemistry. (Edsberg and Oppelstrup (1975).)

n	s_n^0	s_n^1	s_n^2	s_n^3	s_n^4	s_n^5
1	- 1.612(+00)					
2	- 1.259(+00)	8.217(-03)				
3	- 1.066(+00)	3.586(-03)	- 46.17 (-06)			
4	-941.4 (-03)	1.924(-03)	- 15.26 (-06)	252.8 (-09)		
5	-851.9 (-03)	1.168(-03)	- 6.262(-06)	70.69 (-09)	- 1.141(-09)	
6	-783.9 (-03)	770.4 (-06)	- 2.974 (-06)	24.30 (-09)	-302.7 (-12)	10.30 (-12)
7	-729.9 (-03)	539.3 (-06)	- 1.569(-06)	9.690(-09)	- 92.73 (-12)	- 86.16 (-12)
8	-685.7 (-03)	394.7 (-06)	-896.3 (-09)	4.326(-09)	- 87.01 (-12)	- 93.65 (-12)
9	-640.7 (-03)	299.0 (-06)	-544.5 (-09)	2.093 (-09)	69.63 (-12)	- 37.35 (-12)
10	-617.0 (-03)	232.9 (-06)	-347.6 (-09)	1.111(-09)	-176.5 (-12)	- 54.22 (-12)
11	-509.6 (-03)	185.6 (-06)	-231.1 (-09)	500.4 (-12)	20.41 (-12)	1.051(-09)
12	-565.6 (-03)	150.7 (-06)	-159.0 (-09)	335.0 (-12)	182.7 (-12)	-376.3 (-12)
13	-544.2 (-03)	124.3 (-06)	-112.5 (-09)	248.2 (-12)	422.9 (-12)	272.2 (-12)
14	-525.1 (-03)	104.0 (-06)	- 81.63 (-09)	44.73 (-12)	115.6 (-12)	171.6 (-12)
15	-507.9 (-03)	88.04 (-06)	- 60.50 (-09)	132.7 (-12)	175.5 (-12)	146.6 (-12)
16	-492.3 (-03)	75.31 (-06)	- 45.68 (-09)	158.8 (-12)	130.0 (-12)	229.0 (-12)
17	-478.0 (-03)	65.01 (-06)	- 35.06 (-09)	-398.0 (-12)	36.40 (-12)	- 59.43 (-12)
18	-464.9 (-03)	56.58 (-06)	- 57.33 (-09)	764.7 (-12)	- 8.351(-12)	25.55 (-12)
19	-452.9 (-03)	49.61 (-06)	- 21.55 (-09)	-816.2 (-12)	85.82 (-12)	21.17 (-12)
20	-44 1.7 (-03)	43.70 (-06)	- 17.24 (-09)	730.3 (-12)	- 64.16 (-12)	35.18 (-12)
21	-431.3 (-03)	38.86 (-06)	- 13.90 (-09)	-510.8 (-12)	154.0 (-12)	73.30 (-12)
22	-421.6 (-03)	34.69 (-06)	- 11.35 (-09)	264.6 (-12)	46.07 (-12)	180.9 (-12)
23	-412.5 (-03)	31.12 (-06)	- 9.322(-09)	-207.4 (-12)	-577.3 (-12)	-491.8 (-12)
24	-404.0 (-03)	28. C4 (-06)	- 7.729(-09)	-565.3 (-12)	-222.5 (-12)	213.3 (-12)
2s	-396.0 (-03)	25.37 (-06)	- 6.477(-09)	1.185(-09)	34.62 (-12)	-372.1 (-12)
26	-308.5 (-03)	23.05 (-06)	- 5.433(-09)	- 1.131(-09)	910.6 (-12)	- 64.25 (-09)
27	-381.4 (-03)	21.01 (-06)	- 4.628(-09)	6.368(-09)	1.800(-09)	294.2 (-12)
28	-374.6 (-03)	19.21 (-06)	- 3.889(-09)	- 2.019(-09)	4.416(-09)	8.908(-09)
29	-368.2 (-03)	17.63 (-06)	- 3.390 (-09)	14.63 (-09)	6.001(-09)	4.536(-09)
30	-362.1 (-03)	16.22 (-06)	- 2.908(-09)	1.054(-09)	- 1.500 (-09)	-532.3 (-12)
31	-356.3 (-03)	14.96 (-06)	- 2.489(-09)	-900.0 (-12)	-510.3 (-12)	-091.7 (-12)
32	-350.8 (-03)	13.84 (-06)	- 2.172(-09)	-491.7 (-12)	- 1.011 (-09)	-555.8 (-12)
33	-345.5 (-03)	12.83 (-06)	- 1.914(-09)	3.868(-09)	1.174(-09)	-305.7 (-12)
34	-340.5 (-03)	11.92 (-06)	- 1.669(-09)	- 1.159(-09)	- 2.184(-09)	- 1.546(-09)
35	-335.7 (-03)	11.10 (-06)	- 1.515(-09)	- 1.914 (-09)	- 1.497(-09)	-992.2 (-12)
36	-331.0 (-03)	10.35 (-06)	- 1.223(-09)	- 1.220(-09)	- 1.224(-09)	- 1.218(-09)
37	-326.6 (-03)	9.676(-06)	- 1.220(-09)	- 1.224(-09)	- 1.219(-09)	- 1.225(-09)
38	-322.3 (-03)	9.060(-06)	-942.8 (-12)	-958.0 (-12)	-918.2 (-12)	- 1.010(-09)
39	-318.2 (-03)	8.497(-06)	-956.7 (-12)	-945.1 (-12)	- 1.032(-09)	-947.4 (-12)
40	-314.3 (-03)	7.582(-06)	-801.7 (-12)	-765.0 (-12)	-807.0 (-12)	-750.4 (-12)
41	-310.5 (-03)	7.510(-06)	-775.7 (-12)	-810.6 (-12)	-770.6 (-12)	-018.5 (-12)
42	-306.8 (-03)	7.076 (-06)	-4e5.9 (-12)	-665.7 (-12)	-529.1 (-12)	-657.0 (-12)
43	-303.2 (-03)	6.677(-06)	-7e4.5 (-12)	-600.5 (-12)	-746.6 (-12)	-612.1 (-12)
44	-299.8 (-03)	6.308 (-06)	-480.8 (-12)	-466.3 (-12)	-477.1 (-12)	-466.5 (-12)
4s	-296.5 (-03)	5.567(-06)	-469.4 (-12)	-476.3 (-12)	-468.0 (-12)	-464.7 (-12)
46	-293.3 (-03)	5.651(-06)	-480.8 (-12)	-450.5 (-12)	-465.8 (-12)	
47	-290.2 (-03)	5.359(-06)	-501.9 (-12)	-477.7 (-12)		
48	-207.2 (-03)	5.087(-06)	-178.9 (-12)			
49	-204.3 (-03)	4.834(-06)				
50	-201.4 (-03)					

s - formula

n	a_n^0	a_n^1	a_n^2	a_n^3	a_n^4	a_n^5
1	-1.612(+00)					
2	-1.259(+00)	8.217(-03)				
3	-1.066(+00)	3.586(-03)	-46.17(-06)			
4	-941.4(-03)	1.924(-03)	-13.26(-06)	252.3(-09)		
5	-851.9(-03)	1.168(-03)	-6.262(-06)	70.69(-09)	-1.140(-09)	
6	-783.9(-03)	779.4(-06)	-2.974(-06)	24.33(-09)	-303.1(-12)	-7.117(-12)
7	-729.9(-03)	539.3(-06)	-1.569(-06)	9.690(-09)	-101.9(-12)	-17.36(-12)
8	-685.7(-03)	394.7(-06)	-896.3(-09)	4.322(-09)	-46.95(-12)	-8.155(-12)
9	-644.7(-03)	209.0(-06)	-544.5(-09)	2.102(-09)	-25.67(-12)	-19.51(-12)
10	-617.0(-03)	232● 9(-06)	-347.6(-09)	1.095(-09)	-21.29(-12)	-30048(-12)
11	-589.6(-03)	185.4(-06)	-231.1(-09)	601.2(-12)	-11.93(-12)	-19.39(-12)
12	-565.6(-03)	150.7(-06)	-159.0(-09)	345.4(-12)	-32.22(-12)	-16.10(-12)
13	-544.2(-03)	124.3(-06)	-112.5(-09)	201.8(-12)	17.65(-12)	-23.34(-12)
14	-525.1(-03)	104.0(-06)	-81.63(-09)	126.2(-12)	-120.9(-12)	-40.19(-12)
15	-507.9(-03)	R R.04(-06)	-60.50(-09)	70.33(-12)	29.87(-12)	-2.872(-09)
16	-492.3(-03)	75.31(-06)	-45.68(-09)	48.79(-12)	189.9(-12)	97.21(-12)
17	-478.0(-03)	65.01(-06)	-35.07(-09)	22.64(-12)	17.87(-12)	24.45(-12)
18	-464.9(-03)	56.58(-06)	-27.32(-09)	19.07(-12)	23.97(-12)	18.79(-12)
19	-452.9(-03)	49.61(-06)	-21.57(-09)	-3.239(-12)	4.640(-12)	3.399(-12)
20	-441.7(-03)	43.78(-06)	-17.23(-09)	6.601(-12)	3.590(-12)	4.525(-12)
21	-431.3(+03)	38.86(-06)	-13.91(-09)	3.049(-12)	7.711(-12)	4.073(-12)
22	-421.6(-03)	34.69(-06)	-11.34(-09)	-26.33(-12)	-7.766(-12)	-7.533(-12)
23	-412.5(-03)	31.13(-06)	-9.328(-09)	9.232(-12)	-7.554(-12)	-7.788(-12)
24	-404.0(-03)	28.94(-06)	-7.736(-09)	-15.39(-12)	-25.08(-12)	-18.42(-12)
25	-336.0(-03)	25.37(-06)	-6.465(-09)	-23.03(-12)	-15.35(-12)	-7.317(-12)
26	-388.5(-03)	23.05(-06)	-5.440(-09)	21.80(-12)	-11.10(-12)	-10.53(-12)
27	-381.4(-03)	21.01(-06)	-4.607(-09)	-59.82(-12)	-10.65(-12)	-11.13(-12)
28	-374.6(-03)	19.21(-06)	-3.926(-09)	29.35(-12)	-20.12(-12)	-12.86(-12)
29	-368.2(-03)	17.63(-06)	-3.364(-09)	-53.2%(-12)	591.5(-15)	-8.937(-12)
30	-362.1(-03)	16.22(-06)	-2.898(-09)	53.97(-12)	-13.90(-12)	-79.96(-12)
31	-356.3(-03)	14.96(-06)	-2.507(-09)	-76.36(-12)	-25.50(-12)	-14.05(-12)
32	-350.8(-03)	13.84(-06)	-2.181(-09)	-9.763(-12)	61.94(-12)	21.34(-12)
33	-345.5(-03)	12.83(-06)	-1.905(-09)	22.39(-12)	-204.7(-15)	-58006(-12)
34	-340.5(-03)	11.92(-06)	-1.671(-09)	-27.68(-12)	-29.45(-12)	-27.51(-12)
35	-335.7(-03)	11.10(-06)	-1.471(-09)	-28.34(-12)	-27.59(-12)	-28.64(-12)
36	-3.31.0(-03)	10 ● [illegible](-06)	-1.300(-09)	25.32(-12)	-17.76(-12)	-38.82(-12)
37	-326.6(-03)	9.676(-06)	-1.152(-09)	-97.42(-12)	2.787(-12)	-322.9(-15)
38	-322.3(-03)	9.060(-06)	-1.026(-09)	149.5(-12)	-431.4(-15)	2.863(-12)
39	-318.2(-03)	8.497(-06)	-914●2(-12)	-216.6(-12)	34.69(-12)	-78.42(-12)
40	-214.3(-03)	7.982(-06)	-820.6(-12)	240.4(-12)	88.44(-12)	23.15(-12)
41	-313.5(-03)	7.510(-06)	-736.1(-12)	55.42(-12)	763.6(-12)	360.4(-12)
42	-306.8(-03)	7.376(-06)	-661●(-12)	-209.3(-12)	-10.96(-12)	175.5(-12)
43	-303.2(-03)	6.677(-06)	-59Y.2(-12)	280.8(-12)	201.7(-12)	464.9(-12)
44	-299.8(-03)	6.308(-06)	-541.9(-12)	2010 1(-12)	313.5(-12)	233.4(-12)
45	-296.5(-03)	5.967(-06)	-489.6(-12)	-23R.4(-12)	114.0(-12)	229.1(-12)
46	-293.3(-03)	5.651(-06)	-447.9(-12)	762.7(-12)	327.0(-12)	
47	-290.2(-03)	5.359(-06)	-407●9(-12)	156.3(-12)	...	
48	-297.2(-03)	5.087(-06)	-371.1(-12)			
49	-284.3(-03)	4.834(-06)				
50	-281.4(-03)					

a - formula

n	k_n^0	k_n^1	k_n^2	k_n^3	k_n^4	k_n^5
2	518.3 (-03)					
3	510.0 (-03)	2.671(+00)				
4	506.4 (-03)	2.607(+00)	5.051(+00)			
5	504.4 (-03)	2.574(+00)	4.873(+00)	7.499(+00)		
6	503.2 (-03)	2.554(+00)	4.770(+00)	7.273(+00)	-4.270(+00)	
7	502.5 (-03)	2.541(+00)	4.705(+00)	6.552(+00)	-1.938(+00)	-2.034(+00)
8	501.9 (-03)	2.532(+00)	4.662(+00)	12.79 (+00)	1.343(+00)	534.1 (-03)
9	501.6 (-03)	2.526(+00)	4.626(+00)	1.853(+00)	5.001(+00)	-2.326(+00)
10	501.3 (-03)	2.522(+00)	4.613(+00)	-6.171(+00)	-805.0 (-03)	1.373(+00)
11	501.1 (-03)	2.518(+00)	4.588(+00)	-3.372(+00)	-1.129(+00)	2.981(+00)
12	500.9 (-03)	2.515(+00)	4.547(+00)	-1.442(+00)	-603.3 (-03)	4.616(+00)
13	500.8 (-03)	2.513(+00)	4.668(+00)	-306.6 (-03)	1.512(+00)	1.148(+00)
14	500.7 (-03)	2.512(+00)	4.399(+00)	381.3 (-03)	-4.725(+00)	-1.911(+00)
15	500.6 (-03)	2.510(+00)	4.457(+00)	-1.726(+00)	-1.660(+00)	-93.40 (+00)
16	500.5 (-03)	2.509(+00)	6.198(+00)	2.584(+00)	-650.6 (-03)	817.4 (-03)
17	500.5 (-03)	2.508(+00)	2.279(+00)	9.002(+00)	-1.627(+00)	4.602(+00)
18	500.4 (-03)	2.507(+00)	-24.90 (+00)	11.24 (+00)	14.72 (+00)	-2.403(+00)
19	500.4 (-03)	2.506(+00)	1.104(+00)	19.29 (+00)	45.15 (+00)	-2.220(+00)
20	500.4 (-03)	2.506(+00)	-9.150(+00)	-53.59 (+00)	2.822(+00)	29.57 (+00)
21	500.3 (-03)	2.505(+00)	901.3 (-03)	20.58 (+00)	-2.271(+00)	457.0 (-03)
22	500.3 (-03)	2.505(+00)	-8.689(+00)	-220.1 (-03)	442.1 (-03)	1.854(+00)
23	500.3 (-03)	2.505(+00)	-492.2 (+00)	-1.590(+00)	62.69 (-03)	16.21 (+00)
24	500.3 (-03)	2.504(+00)	-245.4 (-03)	2.837(+00)	-1.518(+00)	-2.800(+00)
25	500.2 (-03)	2.504(+00)	-1.612(+00)	-6.142(+00)	-1.015(+00)	973.7 (-03)
26	500.2 (-03)	2.503(+00)	-871.8 (-03)	3.235(+00)	-984.6 (-03)	523.0 (-03)
27	500.2 (-03)	2.504(+00)	-1.110(+00)	-8.296(+00)	-672.2 (-03)	-3.038(+00)
28	500.2 (-03)	2.503(+00)	-963.7 (-03)	3.633(+00)	-1.423(+00)	-1.144(+00)
29	500.2 (-03)	2.503(+00)	-1.044(+00)	619.8 (-03)	421.6 (-03)	-863.9 (-03)
30	500.2 (-03)	2.503(+00)	-1.276(+00)	-3.932(+00)	-6.118(+00)	-2.788(+00)
31	500.2 (-03)	2.502(+00)	-190.2 (-03)	-2.075(+00)	-3.023(+00)	-706.0 (-03)
32	500.1 (-03)	2.502(+00)	-928.7 (-03)	761.2 (-03)	3.815(+00)	-1.267(+00)
33	500.1 (-03)	2.502(+00)	-1.060(+00)	403.9 (-03)	1.294(+00)	910.5 (-03)
34	500.1 (-03)	2.502(+00)	-1.261(+00)	-2.525(+00)	209.9 (-03)	51.99 (+00)
35	500.1 (-03)	2.503(+00)	-529.7 (-03)	1.112(+00)	-2.568(+00)	2.132(+00)
36	500.1 (-03)	2.500(+00)	-1.981(+00)	-2.023(+00)	-1.965(+00)	-2.000(+00)
37	500.1 (-03)	2.504(+00)	180.7 (-03)	-224.4 (-03)	345.0 (-03)	-277.9 (-03)
38	500.1 (-03)	2.499(+00)	-2.612(+00)	-1.580(+00)	-3.558(+00)	-1.523(+00)
39	500.1 (-03)	2.503(+00)	66.99 (-03)	-176.3 (-03)	165.3 (-03)	-204.6 (-03)
40	500.1 (-03)	2.500(+00)	-1.798(+00)	-2.791(+00)	-1.729(+00)	-3.051(+00)
41	500.1 (-03)	2.506(+00)	691.6 (-03)	-365.9 (-03)	421.0 (-03)	-102.4 (-03)
42	500.1 (-03)	2.493(+00)	301.9 (+00)	-1.362(+00)	-13.58 (+00)	-1.546(+00)
43	500.1 (-03)	2.507(+00)	840.2 (-03)	-466.2 (-03)	699.6 (-03)	-314.7 (-03)
44	500.1 (-03)	2.500(+00)	-2.857(+00)	-2.539(+00)	2.527(+00)	
45	500.1 (-03)	2.500(+00)	-1.558(+00)	3.829(+00)		
46	500.1 (-03)	2.499(+00)	-188.2 (-03)			
47	500.1 (-03)	2.510(+00)				
48	500.1 (-03)					

k - estimates

n	t_n^0	t_n^2	t_n^4	t_n^6	t_n^8	t_n^{10}
1	1.612(+00)	-103.3 (-03)	-3.202(-03)	-720.5 (-06)	-66.31(-06)	-6.297(-06)
2	-1.259(+00)	-41.73(-03)	-2.669(-03)	-223.1 (-06)	-20.50(-06)	-1.960(-06)
3	-1.066(+00)	-21.51(-03)	-1.066(-03)	-78.32(-06)	-6.768(-06)	-630.5 (-09)
4	-941.4 (-03)	-12.76 (-03)	-455.9 (-06)	-31.29 (-06)	-2.461(-06)	-216.7 (-09)
5	-851.9 (-03)	-8.285 (-03)	-257.6 (-06)	-13.93(-06)	-983.8 (-09)	-80.49(-09)
6	-783.9 (-03)	-5.736(-03)	-145.5(-06)	-6.759(-06)	-427.4 (-09)	-31.87(-09)
7	-739.9 (-03)	-4.164(-03)	-E7.62(-06)	-3.532(-06)	-199.5(-09)	-14.65(-09)
8	-685.7 (-03)	-3.135(-03)	-55.56 (-06)	-1.955(-06)	-98.85 (-09)	-4.195(-09)
9	-648.7 (-03)	-2.430(-03)	-36.75(-06)	-1.136(-06)	-52.10(-09)	-5.431(-09)
10	-611.0 (-03)	-1.928 (-03)	-23.16(-06)	-680.9 (-09)	-27.77(-09)	-4.034(-09)
11	-589.6 (-03)	-1.561(-03)	-17.75(-06)	-43%.9 (-09)	-16.50(-09)	18.38(-09)
12	-565.6 (-03)	-1.284(-03)	-12.34(-06)	-280.4 (-09)	-10.71 (-09)	-49.67(-09)
13	-544.2 (-03)	-1.072(-03)	-9.490(-06)	-186.8 (-09)	-1.492(-09)	50.49(-09)
14	-525.1 (-03)	-905.2 (-06)	-7.150(-06)	-127.6 (-09)	-11.42(-09)	-93.73(-09)
15	-507.9 (-03)	-772.9 (-05)	-5.479(-06)	-88.37 (-09)	6.711(-09)	86.36 (-09)
16	-4s2.3 (-03)	-666.2 (-06)	-4.261(-06)	-63.55(-09)	-8.695(-09)	-74.41 (-09)
17	-478.0 (-03)	-579.0 (-05)	-3.358(-06)	-435.22(-09)	4.250(-09)	32.27(-09)
18	-464.9 (-03)	-507.0 (-06)	-2.679(-06)	-3.60(-09)	-4.349(-09)	99.13(-09)
19	-452.9 (-03)	-446.3 (-06)	-2.160(-06)	-24.70 (-09)	-2.233(-09)	-352.6 (-09)
20	-411.7 (-03)	-396.4 (-05)	-1.759(-06)	-18.84 (-09)	13.10(-09)	831.5 C-03)
21	-531.3 (-03)	-353.5 (-06)	-1.445(-06)	-14.65 (-09)	-28.93(-09)	-1.767(-06)
22	-421.6 (-03)	-316.8 (-06)	-1.197(-06)	-10.43(-09)	58.50 (-09)	2.995(-06)
23	-412.5 (-03)	-285.2 (-06)	-999.4 (-09)	-9.642(-09)	-110.2 (-09)	-3.489(-06)
24	-404.0 (-03)	-257.9 (-06)	-840.1 (-09)	-4.580(-09)	141.3 (-09)	2.463(-06)
25	-396.0 (-03)	-234.1 (-06)	-710.7 (-09)	-8.617(-09)	-99.72(-09)	-831.2 (-09)
26	-388.5 (-03)	-213.3 (-06)	-604.9 (-09)	-1.919(-09)	22.00(-09)	-388.5 (-09)
27	-381.4 (-03)	-195.0 (-06)	-517.8 (-09)	-3.602(-09)	9.899(-09)	1.336(-06)
28	-374.6 (-03)	-178.8 (-05)	-445.4 (-09)	-3.233(-09)	-27.97(-09)	-282.8 (-09)
29	-3651.2 (-03)	-164.4 (-06)	-385.1 (-09)	-2.379(-09)	55.71(-09)	-7.104(-06)
30	-362.1 (-03)	-151.6 (-05)	-334.4 (-09)	-3.043(-09)	74.88(-09)	19.34(-06)
31	-354.3 (-03)	-140.1 (-05)	-291.7 (-09)	-312.5 (-12)	-395.8 t-091	-28.70(-06)
32	-350.8 (-03)	-129.9 (-05)	-255.5 (-09)	3.476(-09)	599.6 (-09)	34.02(-06)
33	-345.5 (-03)	-120.6 (-05)	-224.6 (-09)	-9.126(-09)	-609.0 (-09)	-40.02(-06)
34	-340.5 (-03)	-112.3 (-06)	-198.1 (-09)	5.030(-09)	681.9 (-09)	48.76(-06)
35	-335.7 (-03)	-10%.7 (-06)	-175.4 (-09)	-7.049(-09)	-773.2 (-09)	-72.66(-06)
36	-331.0 (-03)	-97.83 (-06)	-155.8 (-09)	7.634(-09)	916.3 (-09)	133.9 (-06)
37	-326.6 (-03)	-91.57 (-05)	-138.8 (-09)	-7.132(-09)	-1.677(-06)	-223.1 (-06)
38	-322.3 (-03)	-85.86 (-06)	-124.0 (-09)	10.3s(-09)	3.075(-05)	290.3 (-06)
39	-313.2 (-03)	-PO.64(-05)	-111.1 (-09)	-25.42(-09)	-4.082(-05)	-30102 (-06)
40	-314.3 (-03)	-75.85(-06)	-99.67 (-09)	33.57 (-09)	3.999(-06)	292.5 (-06)
41	-313.5 (-03)	-71.45(-05)	-89.99 (-09)	-31.53(-09)	-3.346(-06)	
42	-303.8 (-03)	-67.40(-06)	-83.98(-09)	22.21(-09)	3.506(-06)	
43	-303.2 (-03)	-63.66(-06)	-73.45(-09)	-19.25(-09)		
44	-293.5 (-03)	-60.21(-05)	-66.49(-09)	30.73(-09)		
45	-296.5 (-03)	-57.01(-06)	-GO.53(-09)			
46	-293.3 (-03)	-54.05 (-05)	-54.87 (-09)			
47	-333.2 (-03)	-51.30(-06)				
48	-287.2 (-03)	-49.74 (-05)				
49	-234.3 (-03)					
50	-281.4 (-03)					

elimination - first method

n	q_n^0	q_n^2	q_n^4	q_n^6	q_n^8	q_n^{10}
1	-1.612(+00)					
2	-1.259(+00)	-60.14 (-03)				
3	-1.066(+00)	-21.55 (-03)	-77.17 (-06)			
4	-941.4 (-03)	-10.46 (-03)	-15.03 (-06)	-1.095(-06)		
5	-851.9 (-03)	-5.979 (-03)	-4.677(-06)	-165.9 (-09)	-16.80 (-09)	
6	-783.9 (-03)	-3.787(-03)	-1.878(-06)	-41.72 (-09)	-1.788(-09)	-337.5 (-12)
7	[illegible] (-03)	-2.575(-03)	-886.1 (-09)	-13.76 (-09)	-334.9 (-12)	-78.61 (-12)
8	-665.7 (-03)	-1.843(-03)	-467.7 (-09)	-5.415(-09)	-111.6 (-12)	[illegible]056(-12)
9	-648.7 (-03)	-1.373(-03)	-269.1 (-09)	-2.421(-09)	-39.91 (-12)	351.1 (-12)
10	-617.0 (-03)	-1.055(-03)	-163.7 (-09)	[illegible]95(-09)	119.2 (-12)	-1.970(-09)
11	-599.6 (-03)	-831.0 (-06)	-105.1 (-09)	[illegible] (-12)	-567.1 (-12)	6.175(-09)
12	-565.6 (-03)	-669.5 (-06)	-70.30 (-09)	-443.6 (-12)	1.347(-09)	-15.13 (-09)
13	-544.2 (-03)	-547.2 (-06)	-43.63 (-09)	-6[illegible]1.06(-12)	-2.704(-09)	26.84 (-09)
14	-525.[illegible] (-03)	-454.7 (-06)	-34.60 (-09)	-383.3 (-12)	3.509(-09)	-25.95 (-09)
15	-507.9 (-03)	-382.6 (-06)	-25.24 (-09)	89.72 (-12)	-1.331(-09)	-13.35 (-09)
16	-492.[illegible] (-03)	-325.6 (-06)	-18.79 (-09)	72.08 (-12)	-5.071 (-09)	105.8 (-09)
17	-478.0 (-03)	-279.8 (-06)	-14.24 (-09)	-561.4 (-12)	13.77 (-09)	-247.1 (-09)
18	[illegible] (-03)	[illegible] (-06)	-11.01 (-09)	762.4 (-12)	-23.02 (-09)	416.9 (-09)
19	-452.9 (-03)	-211.9 (-06)	-8.579(-09)	-1.063(-09)	30.62 (-09)	-571.4 (-09)
20	-441.7 (-03)	-185.4 (-06)	-6.831(-09)	1.037(-09)	-33.33 (-09)	633.9 (-09)
21	-431.3 (-03)	-165.0 (-06)	-5.450(-09)	-891.2 (-12)	27.56 (-09)	-459.6 (-09)
22	-421.6 (-03)	-146.9 (-06)	-4.434(-09)	394.8 (-12)	-5.687(-09)	-137.1 (-09)
23	-412.5 (-03)	-131.4 (-06)	-3.624(-09)	604.0 (-12)	-34.19 (-09)	990.7 (-09)
24	-404.0 (-03)	-119.1 (-06)	-2.966(-09)	-1.627(-09)	58.55 (-09)	-1.231(-06)
25	-396.0 (-03)	-105.7 (-06)	-2.503(-09)	943.3 (-12)	-22.95 (-09)	65.69 (-09)
26	-388.5 (-03)	-96.11 (-06)	-2.095(-09)	959.9 (-12)	-53.11 (-09)	2.050(-06)
27	-381.4 (-03)	-88.00 (-06)	-1.737(-09)	-1.978(-09)	103.9 (-09)	-4.502(-06)
28	-374.6 (-03)	-80.36 (-06)	-1.506(-09)	1.999(-09)	-182.3 (-09)	9.667(-06)
29	-368.2 (-03)	-73.61 (-06)	-1.258(-09)	-4.507(-09)	400.6 (-09)	-19.37 (-06)
30	-362.1 (-03)	-67.62 (-06)	-1.168(-09)	9.868(-09)	-679.3 (-09)	27.98 (-06)
31	-356.3 (-03)	-62.30 (-06)	-352.3 (-12)	-10.67 (-09)	639.3 (-09)	-23.63 (-06)
32	-350.8 (-03)	-57.55 (-06)	-843.1 (-12)	2.708(-09)	-161.1 (-09)	4.446(-06)
33	-345.5 (-03)	-53.29 (-06)	-764.4 (-12)	4.877(-09)	-330.7 (-09)	16.53 (-06)
34	-340.5 (-03)	-49.45 (-06)	-599.0 (-12)	-5.777(-09)	521.7 (-09)	-31.22 (-06)
35	-335.7 (-03)	-46.00 (-06)	-573.9 (-12)	5.301(-09)	-680.4 (-09)	52.15 (-06)
36	-331.0 (-03)	-42.87 (-06)	-453.9 (-12)	-9.612(-09)	1.295(-06)	-108.3 (-06)
37	-326.6 (-03)	-40.03 (-06)	-516.0 (-12)	20.57 (-09)	-2.820(-06)	210.4 (-06)
38	-322.3 (-03)	-37.45 (-06)	-232.7 (-12)	-42.24 (-09)	4.510(-06)	-272.2 (-06)
39	-318.2 (-03)	-35.09 (-06)	-645.1 (-12)	42.10 (-09)	-3.169(-06)	157.5 (-06)
40	-314.3 (-03)	-32.94 (-06)	-350.7 (-12)	14.27 (-09)	-1.161(-06)	20.49 (-06)
41	-310.5 (-03)	-30.97 (-06)	70.43 (-12)	-34.64 (-09)	954.4 (-09)	110.9 (-06)
42	-306.8 (-03)	-29.16 (-06)	-55.33 (-12)	-44.59 (-09)	6.497(-06)	-609.7 (-06)
43	-303.2 (-03)	-27.49 (-06)	-701.3 (-12)	100.2 (-09)	-12.59 (-06)	1.077(-03)
44	-299.8 (-03)	-25.96 (-06)	-11.19 (-12)	-79.56 (-09)	13.46 (-06)	-1.323(-03)
45	-296.5 (-03)	-24.54 (-06)	-372.8 (-12)	81.05 (-09)	-14.74 (-05)	1.479(-03)
46	-293.3 (-03)	-23.23 (-06)	226.3 (-12)	-106.8 (-09)	15.20 (-06)	
47	-290.2 (-03)	-22.01 (-06)	-414.3 (-12)	58.12 (-09)		
48	-287.2 (-03)	-20.98 (-05)	-336.9 (-12)			
49	-234.3 (-03)	-19.83 (-05)				
50	-281.4 (-03)					

elimination - second method

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Appendix.

These program listings are given only to specify exactly how the examples were run, 'not necessarily to show how the extrapolation ought to be coded. In particular, no effort to conserve storage was made. The procedures are written in the language T , which is described in Eric Grosse, "Software restyling in graphics and programming languages", STAN-CS-78-663, 1978.

```
poul(n,p)
   # Poulet recurrence
   integer: n
   real() : p
   real: fn
   case
      7 < n
         fn := n
         p(n) := -(
                  ((((((-fn+17)*fn-116)*fn+415)*fn-849)*fn+978)*fn-504)*p(n-1) +
                  ((((-4*fn+48)*fn-188)*fn+240)*fn)*p(n-2) +
                  ((((2*fn-30)*fn+154)*fn-300)*fn+144)*p(n-3) +
                  ((-fn+7)*fn-9)*p(n-4) +
                  ((-fn+5)*fn-3)*p(n-5)
                  ) / ((((((fn-17)*fn+114)*fn-385)*fn+689)*fn-618)*fn+216)
      7 = n
         p(n) := 2 * 23./(6*5*4*3*2)
      6 = n
         p(n) := 2 *  3./(5*4*3*2)
      5 = n
         p(n) := 2 *  1./(4*3*2)
      n < 5
         p(n) := 0


dpoul(n,p,dp)
   # Poulet recurrence for p(n) -p(n-1)
   procedure: poul
   integer: n
   real() : p, dp
   real: fn
   poul(n,p)
   case
      7 < n
         fn := n
         dp(n) := -(
                  ((((-2*fn+30)*fn-160)*fn+360)*fn-288)*p(n-1) +
                  ((((-4*fn+48)*fn-188)*fn+240)*fn)*p(n-2) +
                  ((((2*fn-30)*fn+154)*fn-300)*fn+144)*p(n-3) +
                  ((-fn+7)*fn-9)*p(n-4) +
                  ((-fn+5)*fn-3)*p(n-5)
                  ) / ((((((fn-17)*fn+114)*fn-385)*fn+689)*fn-618)*fn+216)
      7 = n
         dp(n) := 2 * (23.-6*3) /(6*5*4*3*2)
      6 = n
         dp(n) := 2 * (3.-1*5) /(5*4*3*2)
      5 >= n
         dp(n) := p(n)
```

```
zeta (n,p)
    # zeta
    integer: n
    real(): p
    real: fn
    case
        1 < n
            fn  := n
            p(n)  := p(n-1) + fn**(-1.5)
        1 = n
            p(n) := 1.


dzeta(n,p,dp)
    # zeta
    procedure: zeta
    integer: n
    real(): p , dp
    real: fn
    zeta(n,p)
    case
        1 < n
            fn  := n
            dp(n)  := fn**(-1.5)
        1 = n
            dp(n)  := 1.
```

```
aform(nmax,pn,dpn,h,imax,k,e)
   real(): pn, dpn, h
   integer: nmax,imax, i, n, new
   real(nmax,0:imax): a, s
   real: k, e, af, ab

   next page
   put (*a-formula*)
   next. line
   s := -1 (60)
   a := -1 (60)
   for( 1 <= new <= nmax )
      a (new, 0) := dpn (nev)
      s(new,0) :=pn (new)
      for( 0 <= i <= min ( imax-1, floor ((new-1)/2.)-1 ) )
         n := new-i-l
         af := a(n+1,i) - a(n,i)
         ab := a(n,i) - a(n-1,i)
         case
            af * ab ¬= 0
               a(n,i+1) := a(n,i) * (
                        (  (k+2*i+1.)/(k+2*i)) * a(n,i) * (af-ab)/(af*ab)
                       - ((k+2*i+2.)/(k+2*i)) )
            else
               a(n,i+1):= 0
         case
            a(n+1,i) ¬= a(n,i)
               s(n,i+1) := s(n,i)
                        - (  1 + 1./(k+2*i)) * a(n,i)*a(n+1,i)/(a(n+1,i)-a(n,i))
            else
               s(n,i+1) := s(n,i)
   for( 1 <= n <= nmax )
      put(n,s(n,0)-e,s(n,1)-e,s(n,2)-e,s(n,3)-e,s(n,4)-e,
         s(n,5)-e)
```

```
rich(nmax,pn,dpn,h,imax,k,e)
   real(): pn, dpn, h
   integer: nmax, imax, j, n,  new
   real(nmax,0:2*imax): r
   real: k,  e

   next page
   put('Richardson extrapolation')
   next line
   r := -1(60)
   for( 1 <= new <= nmax )
      r(new,0) := pn(new)
      for(0<=j<= min ( 2*imax-1,  new-2  )   )
         n := new[illegible]
         r(n,j+1) := r(n+1,j) + (r(n+1,j)-r(n,j))
                                  / ( ((n+1.)/(n))**(k+j) - 1. )
   for( 1 <= n <= nmax )
      put(n,r(n,0)-e,r(n,2)-e,r(n,4)-e,r(n,6)-e,r(n,8)-e,
         r(n,10)-e)
```

```
rich2 (nmax,pn,dpn,h,imax,k,e)
   real(): pn, dpn, h
   integer: nmax, imax, i, n,  new
   real(nmax,0:imax): r
   real: k,e, dm1, d0,dp1

   next page
   put('Richardson extrapolation - mark 2')
   next line
   r := -1(60)
   for( 1 <=  new  <= nnax )
      r(new,0) := pn (new)
      for(  0  <= i <= min( imax-1,  floor((new-1)/2.)-1 ) )
         n := new-i-l
         dm1 := (h(n) -h(n+1))/h(n-l)** (k+2*i)
         d0  := (h(n+1)-h(n-1)) / h(n)**(k+2*i)
         dp1 := (h (n-1)-h (c) )/ h(n+1)**(k+2*i)
         r(n,i+1) := (  dm1*r(n-1,i) + d0*r (n,i) + dp1*r(n+1,i)  )
                    / ( dm1+  d0+  dpl  )
   for( 1 <= n <= nmax )
      put(n,r(n,0)-e,r(n,1)-e,r(n,2)-e,r(n,3)-e,r(n,4)-e,
         r(n,5)-e)
```

```
poly(nmax,pn,dpn,h,imax,k,e)
   real(): pn, dpn, h
   integer: nmax,  imax, j, n,  new
   real(nmax,0:2*imax): r
   real: k, e

   next  page
   put('polynomial extrapolation')
   next line
   r := -1(60)
   for( 1 <= new <= nmax )
      r(new,0) := pn(new)
      for( 0 <= j <= min( 2*imax-1, new-2 ) )
         n := new-j-1
         r(n,j+1) := r(n+1,j) + (r(n+1,j)-r(n,j))
                                           / ( h(n)/h(n+j+1) -  1.  )
   for( 1 <= n <= nmax )
      put(n,r(n,0)-e,r(n,2)-e,r(n,4)-e,r(n,6)-e,r(n,8)-e,
         r(n,10)-e)
```

```
rat(nmax,pn,dpn,h,imax,k,e)
   real():pn, dpn, h
   integer: nmax, imax,  j,  n,  new
   real(nmax,-1:2*imax): r
   real: k, e, d

   next page
   put('rational   extrapolation')
   next line
   r := -1(60)
   r(,-1)  := 0
   for( 1 <= neu <= nmax )
      r(new,0)  := pn(new)
      for( 0 <= j <= min (  2*imax-1, new-2 )   )
         n := new-j-l
         case
            r(n+1,j)=r(n+1,j-1)
               r(n,j+1)  := r(n+1,j)
            else
               d := (r(n,j)-r(n+1,j-1)) / (r(n+1,j)-r(n+1,j-1))
               r(n,j+1)  :=  r(n+1,j) + (r(n+1,j)-r(n,j))
                                    / ( d * h(n)/h(n+j+1) -  1.   )
   for( 1 <= n  <= nmax  )
      put(n,r(n,0)-e,r(n,2)-e,r(n,4)-e,r(n,6)-e,r(n,8)-e,
         r(n, 10)-e)




rho(nmax,pn,dpn,h,imax,k,e)
   real(): pn, dpn, h
   integer: nmax,  imdx,  j, n,  new
   real(nmax,-1:2*imax): r
   real:  k,  e

   next. page
   put('rho algorithm*)
   next line
 - r  := -1(60)
   r(,-1) :=  0
   for( 1 <=  new  <= nmax )
      r(new,0)  :=pn (new)
      for(0 <=  j  <=  min  (2*imax-1, new-2 )   )
         n  := new-j-l
         case
            r(n+1,j)=r(n,j)
               r(n,j+1)  :=r(n+1, j-l)
            else
               r(n, j+1) := r(n+1,j-1)  +  (1/h(j+n+1)-1/h(n)) / (r(n+1,j)-r(n,j))
   for(1 <= n <=  nmax  )
      put (n,r(n,0)-e,r(n,2)-e,r(n,4)-e,r(n,6)-e,r(n,8)-e,
         r(n, 10)-e)
```