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ABSTRACT

In this paper we develop a stability theory for the Fourier (or pseudo-spectral) method for linear hyperbolic and parabolic partial differential equations with variable coefficients.

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1. Introduction

The collocation method based on trigonometric interpolation is called the Fourier (or pseudo-spactral) method. It has been used extensively for the computation of approximate solutions of partial differential equations with periodic solutions. A satisfactory theoretical justification for equations with variable coefficients has only existed for equations written in skew symmetric form [3, 6, 7]. Recent work of Majda, McDonough and Osher [8] treats hyperbolic systems with C^∞ coefficients.

In this paper we develop a stability theory for linear hyperbolic and parabolic partial differential equations with variable coefficients. The generalization of these results to nonlinear equations follows if the problem has a sufficiently smooth solution. We restrict our discussion to problems in one space dimension. The extension to problems in more space dimensions is immediate. Error estimates can easily be derived using our results following those in Kreiss and Oliger [7] and Fornberg [3].

2. Trigonometric Interpolation

In this section we collect some known results on trigonometric interpolation (see [4, 5, 7]). Let N be a natural number, $h = (2N+1)^{-1}$, and define grid points $x_\nu = \nu h$, $\nu = 0,1,2,\dots,2N$. Consider a one-periodic function $v(x)$, $v(x) = v(x+1)$, whose values $v_\nu = v(x_\nu)$ are known at the gridpoints x_ν. We define a discrete scalar product and norm by

$$(2.1)\qquad (u(x),v(x))_h = \sum_{\nu=0}^{2N} u(x_\nu)\,\overline{v(x_\nu)}h\ , \quad \|u\|_h^2 = (u,u)_h\ .$$

The trigonometric polynomial $w(x)$ of degree N which interpolates $v(x)$ in the points x_ν, i.e.,

$$(2.2)\qquad w(x_\nu) = v(x_\nu) \qquad \nu = 0,1,2,\dots,2N\ ;$$

is uniquely given by

$$(2.3)\qquad w(x) = \sum_{\omega=-N}^{N} a(\omega)\, e^{2\pi i\omega x}$$

where

$$(2.4)\qquad a(\omega) = (v(x), e^{2\pi i\omega x})_h\ .$$

This follows from the orthonormality of the exponential function,

$$(2.5)\qquad (e^{2\pi inx}, e^{2\pi imx})_h = \begin{cases} 0 & \text{if } 0 < |m-n| \le 2N \\ 1 & \text{if } m=n\ . \end{cases}$$

The usefulness of trigonometric interpolation stems from the fact that the smoothness properties of the function are preserved and that the convergence is rapid for smooth functions. Let the L_2-scalar product and norm be defined by

$$(u,v) = \int_0^1 u\bar{v}\,dx \;, \quad \|u\|^2 = (u,u) \;. \tag{2.6}$$

We will need the following well known theorem.

<u>Theorem 2.1.</u> If w_1, w_2 interpolate v_1 and v_2, respectively, then

$$(w_1,w_2)_h = (w_1,w_2) = (v_1,v_2)_h \quad \text{and} \tag{2.7}$$

$$\|w_1(x)\|^2 = \|v_1(x)\|_h^2 = \sum_{\omega=-N}^{N} |a(\omega)|^2 \tag{2.8}$$

It will be convenient to work with the following class of functions.

<u>Definition 2.1.</u> $P(\alpha,M)$ is the class of all functions $v(x)$ which can be developed in a Fourier series

$$v(x) = \sum_{\omega=-\infty}^{\infty} \hat{v}(\omega) e^{2\pi i\omega x} \tag{2.9}$$

with

$$\sum_{\omega=-\infty}^{\infty} |[|2\pi\omega|^{\alpha} + 1]\hat{v}(\omega)|^2 \le M^2 \;. \tag{2.10}$$

$P(\alpha,M)$ is contained in the Sobelev space H_2^{α}.

We now need the relationship between the Fourier coefficients $\hat{v}(\omega)$

of a given function $v(x)$ and the coefficients $a(\omega)$ of its trigonometric interpolant $w(x)$. This is contained in the following well known result [4, 7].

Theorem 2.2. Let v be given by (2.9) and w given by (2.3) and (2.4) then

$$(2.11) \qquad a(\omega) = \sum_{j=-\infty}^{\infty} \hat{v}(\omega+j(2N+1)) \, , \quad |\omega| \leq N \, .$$

We can now investigate the rate of convergence of the interpolating polynomial to a function $v(x) \in P(\alpha,M)$.

Theorem 2.3. Let $v(x) \in P(\alpha,M)$ with $\alpha > 1/2$. Then

$$(2.12) \qquad \|v(x)-w(x)\| \leq M\left(\frac{1}{(2\pi N)^{2\alpha}} + \frac{2}{(2\pi N)^{2\alpha}}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^{2\alpha}}\right)^{1/2} = \frac{MC_\alpha}{(2\pi N)^\alpha} \, ,$$

where $C_\alpha^2 = 1 + 2 \sum_{j=1}^{\infty} \frac{1}{(2j-1)^{2\alpha}}$.

Proof. We write (2.9) as $v(x) = v_N(x) + v_R(x)$ where

$$v_N(x) = \sum_{\omega=-N}^{N} \hat{v}(\omega)e^{2\pi i\omega x} \, , \quad v_R(x) = \sum_{|\omega|>N} \hat{v}(\omega)e^{2\pi i\omega x} \, .$$

Let $w_N(x)$ and $w_R(x)$ be the trigonometric interpolants of $v_N(x)$ and $v_R(x)$, respectively. They are given by

$$w_N(x) = \sum_{\omega=-N}^{N} a^{(N)}(\omega)e^{2\pi i\omega x}, \quad a^{(N)}(\omega) = (v_N(x),e^{2\pi i\omega x})_h$$

$$w_R(x) = \sum_{\omega=-N}^{N} a^{(R)}(\omega) e^{2\pi i\omega x} , \quad a^{(R)}(\omega) = (v_R(x), e^{2\pi i\omega x})_h \quad .$$

The trigonometric interpolant of $v(x)$ is

$$w(x) = w_N(x) + w_R(x) .$$

$w_N(x)$ interpolates $v_N(x)$ in the $2N+1$ points of (2.2)' and from (2.3) we have

$$w_N(x) = v_N(x) .$$

Therefore,

$$\|v(x)-w(x)\|^2 = \|v_R(x)-w_R(x)\|^2 = \|v_R(x)\|^2 + \|w_R(x)\|^2$$

since $v_R(x)$ is orthogonal to $w_R(x)$. By (2.10) we can write

$$\hat{v}(\omega) = \frac{1}{|2\pi\omega|^{\alpha}+1}$$

where

$$\sum_{\omega=-\infty}^{+\infty} |\tilde{v}(\omega)|^2 \leq M^2 \quad .$$

Therefore,

$$\|v_R(x)\|^2 = \sum_{|\omega| > N} |\hat{v}(\omega)|^2 = \sum_{|\omega| > N} \left| \frac{1}{|2\pi\omega|^{\alpha}+1} \tilde{v}(\omega) \right|^2 \leq \frac{M^2}{(2\pi N)^{2\alpha}} .$$

By Theorem 2.2

$$\|w_R(x)\|^2 = \sum_{\omega=-N}^{N} |a^{(R)}(\omega)|^2 = \sum_{\omega=-N}^{N} \Big| \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \hat{v}(\omega+j(2N+1)) \Big|^2$$

$$= \sum_{\omega=-N}^{N} \left| \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{\tilde{v}(\omega+j(2N+1))}{|2\pi(\omega+j(2N+1))|^{\alpha}+1} \right|^2$$

$$\leq \sum_{\omega=-N}^{N} \left(\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{(|2\pi(\omega+j(2N+1))|^{\alpha}+1)^2} \cdot \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} |\tilde{v}(\omega+j(2N+1))|^2 \right|$$

$$\leq \frac{2M^2}{(2\pi N)^{2\alpha}} \cdot \sum_{j=1}^{\infty} (2j-1)^{-2\alpha}$$

and the theorem follows.

Remark. Observe that the contributions to the error by v_R and w_R are of the same order if $\alpha > 1/2$. w_R is often called the aliasing error. Thus, we see that if v is at all smooth, then aliasing plays no important role.

The following result follows immediately from the last theorem.

Corollary 2.1. Let $v(x) \in P(\alpha,M)$ with $\alpha > j + 1/2$, j a natural number. Then

$$(2.13) \qquad \left\| \frac{d^j}{dx^j} v(x) - \frac{d^j}{dx^j} w(x) \right\|_{II} \leq \frac{MC_{\alpha-j}}{(2\pi N)^{\alpha-j}} .$$

3. Stability of Fourier Methods

Let $v(x)$ be a one-periodic function whose values, $v(x_\nu)$ are known at the gridpoints $x_\nu = \nu h$, $h = (2N+1)^{-1}$. If we want to approximate $dv(x_\nu)/dx$ we can compute the trigonometric interpolant (2.3) of $v(x)$, differentiate it, and use its derivative

$$(3.1) \qquad dw(x_\nu)/dx = \sum_{\omega=-N}^{N} (2\pi i\omega)a(\omega)e^{2\pi i\omega x_\nu}$$

as an approximation of $dv(x_\nu)/dx$. The computation of **(3.1)** in all of the gridpoints $x_\nu, \nu = 0,1,2,\dots,2N$ can be done using two discrete Fourier transforms (DFT) and 2N complex multiplications. Also, if we know that $v(x) \in P(\alpha,M)$ with $\alpha > 3/2$, then Corollary 2.1 gives us the error estimate

$$(3.2) \qquad \|dv/dx - dw/dx\| \leq \frac{MC_{\alpha-1}}{(2\pi N)^{\alpha-1}} .$$

Higher derivatives can be computed analogously.

The above process is linear so it can also be represented using matrix notation. Let

$$\underline{v} = (v(x_0),\dots,v(x_{2N}))' , \quad \underline{y} = (dw(x_0)/dx,\dots,dw(x_{2N})/dx)'$$ [1)]

denote the $(2N+1)$ dimensional vector formed of the grid values of $\underline{v}(x)$ and dw/dx, respectively. Then there is a $(2N+1) \times (2N+1)$

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1) If $\underline{y}$ is a vector then $\underline{y}'$ denotes its transpose and $\underline{y}^*$ its conjugate transpose. The same notation will be used for matrices.

matrix such that

$$\underline{y} = S\underline{v} \tag{3.3}$$

Formulas for the elements of S have been computed by B. Fornberg [2, 3]. He has also shown that S can be considered as the limit of higher and higher order difference approximations.

The scalar product and norm of $\underline{y}$ and $\underline{v}$ are defined by (2.1), i.e.,

$$(\underline{u},\underline{v})_h = \sum^{2N} u(x_\nu)\overline{v(x_\nu)}h \ , \quad \|\underline{u}\|^2 = (\underline{u},\underline{u})_h \ .$$

We need several properties of the operator S. In [6] we proved the following lemma.

<u>Lemma 3.1.</u> S is skew Hermitian, $\|S\|_h = 2\pi N$, the eigenvalues of S are $\lambda_\omega = 2\pi i\omega$, and the corresponding eigenfunctions are

$$\underline{e}_\omega = (1, e^{2\pi i\omega h}, \ldots, e^{2\pi i\omega 2Nh})' \ , \quad \omega = 0, \pm 1, \ldots, \pm N \ .$$

We next consider the approximation of $b(x)\, du/dx$ where $b(x)$ is a smooth one-periodic function. The operator $b(x)\, d/dx$ is essentially skew Hermitean because we can write

$$b(x)\, du/dx = Qu + Ru \tag{3.4}$$

where

$$Qu = \frac{1}{2}(b\,du/dx + d(bu)/dx) , \quad Ru = -\frac{1}{2}\, db/dx\; u .$$

Q is skew Hermitian and R is bounded. There are many problems where $R \equiv 0$. For example, we can write $u\,du/dx$ in the form

$$u\,du/dx = \frac{1}{3}(u\,du/dx + du^2/dx) .$$

Now consider the partial differential equation

$$u_t = b(x)\partial u/\partial x = Qu + Ru , \quad u_t = \partial u/\partial t ,$$

then

$$(u,u)_t = (u,u_t) + (u_t,u) = (u,Qu) + (Qu,u) + (u,Ru) + (Ru,u) = -(u,u\,db/dx) .$$

and we have an energy estimate. If we approximate the above problem by

$$\frac{d\underline{v}}{dt} = \frac{1}{2}(\tilde{b}S + S\tilde{b})\underline{v} - \frac{1}{2}\,\tilde{b}_x\underline{v}$$

where

$$\tilde{b} = \begin{bmatrix} b(x_o) & 0 & \cdots & 0 \\ 0 & b(x_1) & 0 \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & b(x_{2N}) \end{bmatrix} , \qquad \tilde{b}_x = \begin{bmatrix} \frac{db(x_o)}{dx} & 0 & \cdots & 0 \\ 0 & \frac{db(x_1)}{dx} & 0 \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & \frac{db(x_{2N})}{dx} \end{bmatrix}$$

Then we obtain the same energy estimate because

$$(\tilde{b}S + S\tilde{b}) = -(\tilde{b}S + S\tilde{b})^*$$

is skew Hermitian and therefore

$$\frac{d}{dt}(v,v)_h = -(\underline{v},\tilde{D}_x\underline{v})_h$$

The above procedure can be generalized considerably. Consider the parabolic system

$$(3.5) \qquad u_t = (Au_x)_x + Bu_x + Cu \;, \; u_x = \partial u/\partial x$$

where u denotes a vector function with n components, A, B, and C are $n \times n$ matrices, A and B are Hermitian, A is positive definite, and C and $\partial B/\partial x$ are uniformly bounded. We can rewrite this system in the form

$$(3.6) \qquad u_t = (Au_x)_x + \frac{1}{2}(B\,u_x + (B\,u)_x) + C_1 u \;,$$

where

$$C_1 = C - \frac{1}{2}\,\partial B/\partial x \;.$$

We then obtain the energy estimate

$$(u,u)_t = -2(u_x,A\,u_x) + 2 \text{ Real } (u,C_1 u)$$

which depends solely on the property that $\partial/\partial x$ is skew Hermitian. Thus, we obtain a corresponding estimate if we replace $\partial/\partial x$ by S and approximate (3.6) by

$$(3.7) \qquad \frac{d\underline{v}}{dt} = \tilde{S}\,\tilde{A}\tilde{S}\underline{v} + \frac{1}{2}(\tilde{B}\,\tilde{S} + \tilde{S}\tilde{B})\underline{v} + \tilde{C}_1\underline{v}$$

The estimate is

$$\frac{d}{dt}\|\underline{v}\|_h^2 \le (\underline{v},(\tilde{C}_1 + \tilde{C}_1^*)\underline{v})_h$$

where we extend our earlier definitions of the discrete norm and inner product in the obvious way. Here v is the vector with vector components $v(x_\nu)$ and $\tilde{A}$, $\tilde{B}$, $\tilde{C}_1$, and $\tilde{S}$ are block diagonal matrices with blocks $A(x_\nu)$, $B(x_\nu)$, $C_1(x_\nu)$, and S, respectively.

The system of ordinary differential equations **(3.7)** can be solved using an appropriate difference method for ordinary differential equations. However, the approximation (3.7) requires about twice as much work as the simpler approximation

(3.8) $$\frac{d\underline{v}}{dt} = \tilde{S}\tilde{A}\tilde{S}\underline{v} + \tilde{B}\tilde{S}\underline{v} + \tilde{C}\underline{v}$$

of (3.5). Since numerical experience has shown that approximations of the form **(3.8)** can be unstable, it is desirable to find ways of stabilizing them which are cheaper to use than reverting to **(3.7).** We can achieve this by adding appropriate dissipative or projective operators. We will now develop this approach in detail.

It is easier to do this if we work within the space T_N of trigonometric polynomials

(3.9) $$p(x) = \sum_{\omega=-N}^{N} \hat{p}(\omega)e^{2\pi i\omega x}$$

A vector function $v(x)$ or a matrix function $B(x)$ will belong to T_N if all their components do. There is a one-to-one correspondence between a polynomial **(3.9)** and its values

$$\underline{v} = (v(x_o),\ldots,v(x_{2N}))' .$$

Thus, there is a linear operator P such that

$$P\underline{v} = v(x) , \quad \text{i.e.,} \quad v(x_\nu) = v_\nu , \quad \nu = 0,1,2,\ldots,2N.$$

If $v(x) \in T_N$ then

$$\text{(3.10)} \qquad PS\underline{v} = dv/dx .$$

Let $B(x), v(x) \in T_N$. Then we define $w(x) = B(x)*v(x)$ to be the convolution

$$\text{(3.11)} \qquad w(x) = B(x)*v(x) = \sum_{\nu=-N}^{N} \hat{w}(\nu)e^{2\pi i\nu x}$$

with

$$\text{(3.12)} \quad \hat{w}(\nu) = \begin{cases} \sum_{\mu=-N}^{+N} \hat{B}(\mu)(\hat{v}(\nu-\mu) + \hat{v}(\nu-2N-1-\mu)) & \text{for} \quad \nu \geq 0 \\ \sum_{\mu=-N}^{N} \hat{B}(\mu)(\hat{v}(\nu-\mu) + \hat{v}(\nu+2N+1-\mu)) & \text{for} \quad \nu < 0 . \end{cases}$$

where we have used the convention that $\hat{v}(\omega) = \hat{B}(\omega) = 0$ if $|\omega| > N$. $B(x)v(x)$ is a trigonometric polynomial of order 2N. By theorem 2.2 its interpolant is given by $B(x)*v(x)$. Therefore,

$$\text{(3.13)} \qquad w(x) = P(\tilde{B}\underline{v}) = B(x)*v(x) .$$

Lemma 3.2. Let $B(x) \in T_N$ be a matrix and $v,w \in T_N$ be vector functions. Then

$$|(w,B*v)| \leq \max_{0 \leq x \leq 1} |B(x)| \cdot \|w\| \, \|v\| .$$

and, if B is Hermitian,

$$(w,B*v) = (B*w,v) .$$

Proof. By theorem 2.1 and (3.13)

$$(w,B*v) = (w,B*v)_h = (\underline{w},\tilde{B}\underline{v})_h .$$

If B is Hermitian, then

$$(\underline{w},\tilde{B}\underline{v})_h = (\tilde{B}\underline{w},\underline{v})_h = (B*w,v)_h = (B*w,v) .$$

Also,

$$|(\underline{w},\tilde{B}\underline{v})_h| \leq |\tilde{B}| \, \|\underline{w}\|_h \|\underline{v}\|_h = \max_{0 \leq x_\nu \leq 1} |B(x_\nu)| \, \|w\| \, \|v\| .$$

and the lemma is proved.

We can now write equation **(3.8)** as an evolution equation in T_N via the isomorphiam P.

(3.14) $$v_t = (A_N*v_x)_x + B_N*v_x + C_N*v$$

where A_N, B_N, C_N and v are the trigonometric polynomials in T_N which interpolate the discrete values $A(x_\nu)$, $B(x_\nu)$, $C(x_\nu)$, $v(x_\nu)$, respectively. The term $w = B_N*v_x$ can be written as

$$w = B_N*v_x = Qv + Rv$$

where

$$(3.15)\quad \begin{aligned} Qv &= \tfrac{1}{2}(B_N * v_x + (B_N * v)_x)\ , \\ Rv &= \tfrac{1}{2}(B_N * v_x - (B_N * v)_x)\ . \end{aligned}$$

It follows from lemma 3.2 that the operator Q is skew Hermitian.

Straightforward application of (3.12) gives us

$$Rv = R_1 v + R_2 v\ , \quad R_j v = \sum_{\omega=-N}^{N} \hat{r}_j\ We^{2\pi i\omega x}\ , \quad j = 1,2,$$

where

$$(3.16)\quad \hat{r}_1(\omega) = -\pi i \begin{cases} \displaystyle\sum_{\mu=-N}^{N} \mu \hat{B}_N(\mu)\,(\hat{v}(\omega-\mu) + \hat{v}(\omega-2N-1-\mu)) & \text{for} \quad \omega \geq 0 \\ \displaystyle\sum_{\mu=-N}^{+N} \mu \hat{B}_N(\mu)\,(\hat{v}(\omega-\mu) + \hat{v}(\omega+2N+1-\mu)) & \text{for} \quad \omega < 0 \end{cases}$$

$$(3.17)\quad \hat{r}_2(\omega) = \pi i(2N+1) \begin{cases} \displaystyle -\sum_{\mu=-N}^{N} \hat{B}_N(\mu)\hat{v}(\omega-2N-1-\mu) & \text{for} \quad \omega \geq 0 \\ \displaystyle\sum_{\mu=-N}^{+N} \hat{B}_N(\mu)\hat{v}(\omega+2N+1-\mu) & \text{for} \quad \omega < 0\ . \end{cases}$$

By (3.12)

$$(3.18)\qquad R_1 v = -\tfrac{1}{2}\, dB_N/dx * v\ .$$

Therefore, by lemma 3.2, the operator R_1 is bounded if $B \in P(\alpha,M)$ with $\alpha > 3/2$, certainly if B is twice continuously differentiable (see [1]).

In general we can not expect that $(v,R_2 v)$ is bounded independent of N. For example, if $B(x) = I(1 + \frac{1}{2}\sin 2\pi x)$ then

$$\hat{B}_N(0) = I,\ \hat{B}_N(1) = -\hat{B}_N(-1) = -\frac{i}{4} I,\ \hat{B}_N(\omega) = 0 \text{ if } |\omega| \neq 0,1$$

and

$$\hat{r}_2(N) = \frac{\pi}{4}(2N+1)\hat{v}(-N)\ ,\ \hat{r}_2(-N) = \frac{\pi}{4}(2N+1)\hat{v}(N)\ ,\ \hat{r}_2(\omega) = 0 \text{ if } |\omega| \neq N\ .$$

Therefore, [illegible] Parseval's [illegible]

$$(v,R_2v) = \frac{\pi}{2}(2N+1)\,\text{Real}\{\hat{v}(N)\overline{\hat{v}(-N)}\}\ .$$

Now assume that there are constants M_1 and $\beta > 1$, independent of N, such that

$$|\hat{B}_N(\mu)| \leq \frac{M}{|2\pi\mu|^{\beta}} \quad \text{for} \quad \mu \neq 0\ . \tag{3.19}$$

Then we obtain

$$\begin{aligned} |(v,R_2v)| \leq\ & \pi(2N+1)\Big(\Big|\sum_{\omega=0}^{N} \hat{v}(\omega)\ \overline{\sum_{\mu=-N}^{N} \hat{B}_N(\mu)\hat{v}(\omega-2N-1-\mu)}\Big| \\ & + \Big|\sum_{\omega=-N}^{-1} \hat{v}(\omega)\ \overline{\sum_{\mu=-N}^{N} \hat{B}_N(\mu)\hat{v}(\omega+2N+1-\mu)}\Big|\Big) \end{aligned} \tag{3.20}$$

where $\hat{v}(\tau) = 0$ for $|\tau| > N$. By (3.19)

$$\Big|\sum_{\omega=0}^{N} \hat{v}(\omega)\ \overline{\sum_{\mu=-N}^{N} \hat{B}_N(\mu)\hat{v}(\omega-2N-1-\mu)}\Big| \leq$$

$$M_1 \sum_{\substack{\mu=-N \\ \mu\neq 0}}^{N} \frac{1}{|2\pi\mu|^{\beta}} \sum_{\omega=0}^{N} |\hat{v}(\omega)|\ |\hat{v}(\omega-2N-1-\mu)| \leq$$

$$M_1 \sum_{\mu=-N}^{-1} \frac{1}{|2\pi\mu|^{\beta}} \cdot \sum_{\omega=N+\mu+1}^{N} |\hat{v}(\omega)|\ |\hat{v}(\omega-2N-1-\mu)| \leq$$

$$\frac{M_1}{2} \sum_{\mu=-N}^{-1} \frac{1}{|2\pi\mu|^{\beta}} \sum_{\omega=N+\mu+1}^{N} \left(|\hat{v}(\omega)|^2 + |\hat{v}(\omega-2N-1-\mu)|^2\right) \leq$$

$$\frac{M_1}{2} \sum_{\mu=-N}^{-1} \frac{1}{|2\pi\mu|^{\beta}} \sum_{\omega=N+\mu+1}^{N} \left(|\hat{v}(\omega)|^2 + |\hat{v}(-\omega)|^2\right) \leq$$

$$\frac{M_1}{2} \sum_{\substack{\omega=-N \\ \omega\neq 0}}^{N} |\hat{v}(\omega)|^2 \sum_{\mu=N-|\omega|+1}^{N} \frac{1}{(2\pi\mu)^{\beta}}$$

There is a constant K_1 such that

$$\sum_{\mu=N-|\omega|+1}^{N} \frac{1}{(2\pi\mu)^{\beta}} \leq \frac{K_1}{(N-|\omega|+1)^{\beta-1}} \quad ,$$

$K_1 = (1/2\pi)^{\beta}(\beta/(\beta-1))$ will do. Furthermore, the same estimate holds for the second sum on the right side of (3.20). We obtain

(3.21)
$$|(v,R_2v)| \leq M_1K_1 \cdot \sum_{\omega=-N}^{N} \gamma_{\omega}|\hat{v}(\omega)|^2 \quad ,$$

where

$$\gamma_{\omega} = \frac{(2N+1)\pi}{(N-|\omega|+1)^{\beta-1}} \qquad \text{if} \quad \omega \neq 0 \ , \ \gamma_o = 0 \quad .$$

Consider the system (3.14). We have, Using (3.15) and (3.18),

(3.22)
$$(v,v)_t = 2 \text{ Real } \{(v,(A_N*v_x)_x) + (v,Qv) + (v,Rv) + (v,C_N*v)\}$$

$$= -2(v_x,A_N*v_x) + 2 \text{ Real } (v,(C_N - \tfrac{1}{2}\,\partial B_N/\partial x) * v) + 2 \text{ Real } (v,R_2v) \ .$$

A is positive definite by assumption, i.e., there is a constant $\sigma > 0$ such that $A \geq \sigma I$. Therefore,

$$(v_x, A_N * v_x) = (\underline{v_x}, \tilde{A}\underline{v_x})_h \geq \sigma \|v_x\|^2$$

By Parseval's relation and (3.21)

$$-2(v_x, A_N * v_x) + 2 \text{ Real } (v, R_2 v) \leq$$

(3.23)

$$2 \sum_{\omega=-N}^{N} (-\sigma(2\pi\omega)^2 + M_1 K_1 \gamma_\omega) |v(\omega)|^2 \leq 2\alpha \|v\|^2, \quad \alpha = \max_{0 < |\omega| \leq N} (-\sigma(2\pi\omega)^2 + M_1 K_1 \gamma_\omega).$$

Since $\sigma > 0$, and if $\beta > 2$, then α is bounded independent of N, and (3.22) and lemma 3.2 give us the energy estimate

$$(v,v)_t \leq 2 \text{ Real } (v, (C_N - \tfrac{1}{2} \partial B_N/\partial x) * v) + 2\alpha \|v\|^2$$

$$\leq 2(\max_x |C_N - \tfrac{1}{2} \partial B_N/\partial x| + \alpha) \|v\|^2 .$$

If $\beta \geq 3$ then a simple calculation gives us

$$\gamma_\omega \leq \frac{(2N+1)\pi}{(N-|\omega|+1)^2} \leq 2\pi(1 + \tfrac{1}{N}) \frac{|\omega|^2}{N} .$$

Therefore, if $2\pi\sigma \geq M_1 K_1 (N^{-1} + N^{-2})$ then α in (3.23) is nonpositive and we obtain the following theorem from (3.22).

<u>Theorem 3.1.</u> If $\beta \geq 3$ and $2\pi\sigma > (M_1 K_1 (N^{-1} + N^{-2})$, then the solutions of (3.14) satisfy the estimate

(3.24) $$(v,v)_t \leq 2 \text{ Real } (v,(C_N - \tfrac{1}{2}\, \partial B_N/\partial x) * v) \quad .$$

This is entirely satisfactory since it is essentially the same as the corresponding estimate for the differential equation. Furthermore, N can always be chosen large enough so that $2\pi\sigma > M_1K_1(N^{-1} + N^{-2})$, at least in principle.

For hyperbolic equations, A "0, the situation is not as good. In this case we have to control the smoothness of v. Experience has shown that higher frequency modes can grow if this is not done.

Let $m > 1$ be a natural number,

$$v = \sum_{\omega=-N}^{N} \hat{v}(\omega)e^{2\pi i\omega x}$$

and define v_1, v_2 by

(3.25) $$v_1 = \sum_{|\omega| \leq N_1} \hat{v}(\omega)e^{2\pi i\omega x} \quad , \qquad v_2 = v - v_1 \quad ,$$

where $N_1 = (1 - 1/m)N$. The smoothing operator $H = H(j,m,D)$ mapping T_N into T_N is defined by

(3.26) $$w = Hv = \sum_{\omega=-N}^{N} \hat{w}(\omega)e^{2\pi i\omega x}$$

where

$$\hat{w}(\omega) = \begin{cases} \hat{v}(\omega) & \text{if } |\omega| \leq (1 - \frac{1}{m})N \\ \hat{v}(\omega) & \text{if } |\omega| > (1 - \frac{1}{m})N \text{ and } |\hat{v}(\omega)| \leq \dfrac{D\|v_1\|}{(2\pi|\omega|)^j} \\ \dfrac{D\|v_1\|}{(2\pi|\omega|)^j} \dfrac{\hat{v}(\omega)}{|\hat{v}(\omega)|} & \text{otherwise .} \end{cases}$$

j is a natural number and D is a constant. Thus, only the higher frequencies are modified, i.e.,

$$Hv_1 = v_1 , \qquad \|Hv\| \leq \|v\| .$$

We want to show that H is a very mild form of smoothing.

Lemma 3.3. Let $\gamma > 0$ be a constant and j a natural number. Consider the class of functions with

$$\|\partial^j u/\partial x^j\|^2 \leq \gamma^2 \|u\|^2 . \tag{3.27}$$

If

$$\left(2\pi N\left(\frac{m-1}{m}\right)\right)^{2j} \geq 2\gamma^2 \text{ and } D \geq \sqrt{2}\,\gamma \tag{3.28}$$

then

$$Hu = u .$$

Proof. Let $u \in T_N$ and write it in the form

$$u = u_1 + u_2 \quad \text{where} \quad \hat{u}_1(\omega) = 0 \quad \text{for } |\omega| > \frac{m-1}{m} N$$

(3.27) implies

$$\left(\frac{m-1}{m} 2\pi N\right)^{2j} \|u_2\|^2 \leq \|\partial^j u_2/\partial x^j\|^2 \leq \gamma^2 (\|u_1\|^2 + \|u_2\|^2).$$

By (3.28)

$$\|u_2\|^2 \le \|u_1\|^2 .$$

Therefore, for $\omega \neq 0$,

$$|\hat{u}(\omega)|^2 < \gamma^2(2\pi|\omega|)^{-2j}\|u\|^2 \le 2\gamma^2(2\pi|\omega|)^{-2j}\|u_1\|^2$$

and the lemma follows.

Instead of (3.14) we now consider the approximation

$$(3.29) \qquad \frac{\partial v}{\partial t} = B_N * Hv_x + C_N * v ,$$

To see that (3.29) has a unique solution we need.

Lemma 3.4. H is a Lipschitz continuous operator from T_N into TN.

Proof. Let $v^{(i)} \in$ TN and $w^{(i)} = Hv^{(i)}$, i = 1,2. Note that $|\hat{w}^{(i)}(\omega)| \le |\hat{v}^{(i)}(\omega)|$ and $\arg \hat{w}^{(i)}(\omega) = \arg \hat{v}^{(i)}(\omega)$, i = 1,2, both follow from the definition of H. Consider the quantities $|\hat{w}^{(1)}(\omega) - \hat{w}^{(2)}(\omega)|$. We consider three cases. Let

$$J_1 = \{\omega \,|\, |\omega| \le N,\ \hat{w}^{(\ell)}(\omega) = \hat{v}^{(\ell)}(\omega),\ \ell = 1,2\}$$

$$J_2 = \{\omega \,|\, |\omega| < N,\ \hat{w}^{(\ell)}(\omega) \neq \hat{v}^{(\ell)}(\omega),\ \ell = 1,2\}$$

$$J_3 = \{\omega \,|\, |\omega| \le N,\ \omega \notin J_1 \cup J_2\}$$

From the definition of H it follows that $\omega \in J_1$ if $|\omega| \le N_1 = N(1-1/m)$. If $\omega \in J_1$, then $|\hat{w}^{(1)}(\omega) - \hat{w}^{(2)}(\omega)| = |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)|$. If $\omega \in J_2$, then

$$d(\omega) \equiv |\hat{w}^{(1)}(\omega) - \hat{w}^{(2)}(\omega)| = \left|K(\omega)\|v_1^{(1)}\| \frac{\hat{v}^{(1)}(\omega)}{|\hat{v}^{(1)}(\omega)|} - K(\omega)\|v_1^{(2)}\| \frac{\hat{v}^{(2)}(\omega)}{|\hat{v}^{(2)}(\omega)|}\right|$$

where $K(\omega) = D/(2\pi|\omega|)^j$. We assume, without loss of generality, that $\|v_1^{(1)}\| > \|v_1^{(2)}\|$. Using the triangle inequality we obtain

$$d(\omega) \leq \left| K(\omega)\|v_1^{(1)}\| \frac{\hat{v}^{(1)}(\omega)}{|\hat{v}^{(1)}(\omega)|} - K(\omega)\|v_1^{(2)}\| \frac{\hat{v}^{(1)}(\omega)}{|\hat{v}^{(1)}(\omega)|} \right| +$$

$$\left| K(\omega)\|v_1^{(2)}\| \frac{\hat{v}^{(1)}(\omega)}{|\hat{v}^{(1)}(\omega)|} - K(\omega)\|v_1^{(2)}\| \frac{\hat{v}^{(2)}(\omega)}{|\hat{v}^{(2)}(\omega)|} \right| ,$$

We can bound the first term of our last expression by

$$K(\omega)\big| \|v_1^{(1)}\| - \|v_1^{(2)}\| \big| \leq K(\omega)\|v_1^{(1)} - v_1^{(2)}\| \leq K(\omega)\|v^{(1)} - v^{(2)}\|$$

since the two complex numbers have equal arguments. We can bound the second term by $|\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)|$ utilizing the triangle inequality and the fact that the distance between two points $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$ is a non-decreasing function of r_1 if $r_1 \geq r_2$. Finally, we obtain

$$(3.30) \qquad d(\omega) < K(\omega)\|v^{(1)} - v^{(2)}\| + |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)|$$

if $\omega \in J_2$. Let $\omega \in J_3$ and assume without loss of generality that $\hat{v}^{(1)}(\omega) \neq \hat{w}^{(1)}(\omega)$ and $\hat{v}^{(2)}(W) = \hat{w}^{(2)}(\omega)$. If $|\hat{v}^{(2)}(\omega)| > K(\omega)\|v_1^{(1)}\|$, then

$$d(\omega) \leq \left|\hat{w}^{(1)}(\omega) - K(\omega)\|v_1^{(1)}\| \frac{\hat{v}^{(2)}}{|\hat{v}^{(2)}|}\right| + \left|K(\omega)\|v_1^{(1)}\| \frac{\hat{v}^{(2)}}{|\hat{v}^{(2)}|} - \hat{v}^{(2)}(\omega)\right|$$

$$\leq |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)| + \left|K(\omega)\|v_1^{(1)}\| \frac{\hat{v}^{(2)}}{|\hat{v}^{(2)}|} - K(\omega)\|v_1^{(2)}\| \frac{\hat{v}^{(2)}}{|\hat{v}^{(2)}|}\right|$$

$$\leq |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)| + K(\omega) \, | \, \|v_1^{(1)}\| - \|v_1^{(2)}\| \, |$$

$$\leq |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)| + K(\omega)\|v^{(1)} - v^{(2)}\|$$

If $|\hat{v}^{(2)}(\omega)| \leq K(\omega)\|v_1^{(1)}\|$, then it easily follows that $d(\omega) < |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)|$. Thus, if $\omega \in J_3$, $d(\omega)$ satisfies the inequality (3.30). Now we estimate

$$\|w^{(1)} - W^{(2)}\|^2 = \sum_{\omega=-N}^{N} d^2(\omega)$$

$$< \sum_{\omega \in J_1} |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)|^2$$

$$+ \sum_{\omega \in J_2 \cup J_3} (K(\omega)\|v^{(1)} - v^{(2)}\| + |\hat{v}^{(1)}(\omega) - \hat{v}^{(2)}(\omega)|)^2$$

$$< (2 + 4K^2(N_1)(N-N_1))\|v^{(1)} - v^{(2)}\|^2$$

which yields the desired result.

From Lemma 3.4 it follows that the operator on the right hand side of (3.29) is Lipschitz continuous and it then follows that (3.29), with initial data, has a unique solution. $v(t)$. We will now derive estimates for the norm of this solution.

We have

$$\partial_t \|v\|^2 = 2 \text{ Real } (v, v_t) = 2 \text{ Real } (v, B_N * Hv_x + C_N * v) .$$

The term $(v, C_N * v)$ is easily bounded as before using Lemma 3.2 if $C \in P(\alpha, M)$ with $\alpha > 1/2$, or is continuously differentiable.

We write

$$(v, B_N * Hv_x) = (v, B_N * (v_1)_x) + (v, B_N * ((v_1)_x - Hv_x)$$

splitting $v = v_1 + v_2$ and utilizing the fact that H does not alter the first N_1 Fourier components of the vector it operates on. We then further split $_{4v} * (v_1)_x$ in terms of Q and $R = R_1 + R_2$ as before to obtain

$$2 \text{ Real } (v, B_N * Hv_x) = 2 \text{ Real } \{(v, R_1 v_1) + (v, R_2 v_1) + (v, B_N * ((v_1)_x - Hv_x)\}$$

where we have used the fact the Q is skew-hermitian. Recall that $R_1 v_1 = -\frac{1}{2} dB_N/dx * v_1$ which is bounded as before if $B \in P(\alpha, M)$ with $\alpha > 3/2$. We have

$$(3.31) \quad \partial_t \|v\|^2 = 2 \text{ Real } (v, C_N * v - \tfrac{1}{2} dB_N/dx * v_1) + 2 \text{ Real } (v, R_2 v_1) + 2 \text{ Real } (v, B_N * ((v_1)_x - Hv_x)$$

the first term is bounded and converges to the proper estimate for the differential equation. We will now construct bounds for the last two terms. We assume that B_N satisfies (3.19) and obtain, corresponding to (3.20) ,

$$(3.32) \quad |(v, R_2 v_1)| < \pi(2N+1)\Big(\Big|\sum_{\omega=0}^{N} \hat{v}(\omega) \overline{\sum_{\mu=-N}^{N} \hat{B}_N(\mu)\hat{v}_1(\omega-2N-1-\mu)}\Big| + \Big|\sum_{\omega=-N}^{-1} \hat{v}(\omega) \overline{\sum_{\mu=-N}^{N} \hat{B}_N(\mu)\hat{v}_1(\omega+2N+1-\mu)}\Big|\Big) .$$

Utilizing (3.19) we obtain

$$\left| \sum_{\omega=0}^{N} \hat{v}(\omega) \overline{\sum_{\mu=-N}^{N} \hat{B}_N(\mu)\hat{v}_1(\omega-2N-1-\mu)} \right| \leq$$

$$M_1 \sum_{\omega=1}^{N} |\hat{v}(\omega)| \sum_{\mu=-N}^{\omega-2N-1+N_1} |2\pi\mu|^{-\beta} |\hat{v}_1(\omega-2N-1-\mu)| \leq$$

$$M_1 \left(\frac{m}{2\pi N}\right)^{\beta} \sum_{\omega=1}^{N} |\hat{v}(\omega)| \sum_{\mu=-N}^{\omega-2N-1+N} |\hat{v}_1(\omega-2N-1-\mu)| \leq$$

$$M_1 \left(\frac{m}{2\pi N}\right)^{\beta} N^{1/2} \|v_1\| \sum_{\omega=1}^{N} |\hat{v}(\omega)| \leq$$

$$M_1 \left(\frac{m}{2\pi N}\right)^{\beta} N \|v_1\| \|v\|$$

and the second term on the right hand side of (3.32) also satisfies the same estimate. We obtain

$$|(v,R_2v_1)| \leq 2\pi M_1(2N^2+N)\left(\frac{m}{2\pi N}\right)^{\beta} \|v_1\| \|v\| \tag{3.33}$$

$$\leq (3/(2\pi)^{\beta-1}) M_1 m^{\beta} N^{-\beta+2} \|v_1\| \|v\|$$

We only have the term $(v, B_N * ((v_1)_x - Hv_x)$ left to estimate. We have, via lemma 3.2, that

$$|(v,B_N * ((v_1)_x - Hv_x)| \leq \max_x |B_N| \|v\| \|((v_1)_x - Hv_x)\| . \tag{3.34}$$

From the definition of H we have

$$\|(v_1)_x - Hv_x)\| \le \frac{D\|(v_1)_x\|}{(2\pi)^j} \sum_{\omega=N_1+1}^{N} \frac{1}{|\omega|^j}$$

(3.35)

$$\le \frac{2D}{(2\pi)^{j-1}(j-1)} N_1^{2-j} \|v\|$$

if $j \ge 2$.

We can now collect our estimates (3.31), (3.33), (3.34) and (3.35) to obtain

Theorem 3.2. Let $j = \beta \ge 2$, then the solutions of (3.29) satisfy the estimate

$$\partial_t \|v\|^2 \le 2 \text{ Real } (v, C_N * v - \tfrac{1}{2} dB_N/dx * v_1) +$$

(3.36)

$$[(6/(2\pi)^{\beta-1})M_1 m^\beta N^{2-\beta} + (4D/(2\pi)^{j-1}(j-1))(\tfrac{m}{m-1})^{j-2} N^{2-j} \max_x |B_N|]\|v\|^2.$$

If $j = \beta > 2$, then the estimate (3.35) converges to the corresponding estimate for the differential equation as $N \to \infty$.

If the coefficients are smooth the estimate (3.35) is quite satisfactory for sufficiently large N. We have been able to obtain this estimate by introducing the smoothing operator H and by requiring that the coefficients C and B be smooth. A similar estimate can be obtained, with much less effort, if we were to alter the definition of H such that $\hat{w}(\omega) = 0$ if $|\omega| > N_1$, or $\hat{w}(\omega) = \hat{v}(\omega)/((2\pi[|\omega|-N_1]_+)^j+1)$ if $|\omega| > N_1$ where $[g]_+$ denotes the positive part of g. These are both linear operators. However, the resulting methods are less accurate.

Convergence estimates can be constructed utilizing the estimates of theorems 3.1 and 3.2 following those of Kreiss and Oliger [7] and

Fornberg [3] and the approximation results of Bube [1].

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