CS109A Notes for Lecture 2/12/96

Assignments With Replacements

- We are given n "items," to each of which we must assign one of k "values."
  - □ Each value may be used any number of times from 0 up.
  - □ Let $W(n, k)$ be the number of ways.
- How many different ways may we assign values to the items?
  - □ "Different ways" means that one or more of the items get different values.

Basis: $W(1, k) = k$ (assign the one item any of k values.

Induction: With $n + 1$ items, assign the first in k ways, and the remaining n in $W(n, k)$ ways.

- Recurrence:

  $W(1, k) = k$
  $W(n + 1, k) = k \times W(n, k)$

- k is "carried along"; induction is really on n, as if it were

  $T(1) = k$
  $T(n) = k \times T(n - 1)$

- Easy solution: $T(n) = k^n$.

Example: Are there more strings of length 5 built from three symbols or strings of length 3 built from five symbols?

- Consider strings of length 5 whose positions are chosen independently from symbols 0, 1, and 2.
  - □ Values = $\{0, 1, 2\}$ and "items" = the five positions.
  - □ Number of strings = $3^5 = 243$.

- Consider strings of length 3 with positions chosen independently from symbols 0, 1, 2, 3, and 4.
  - □ Values = $\{0, 1, 2, 3, 4\}$; "items" = three positions.
  - □ Number of strings = $5^3 = 125$.

Permutations

Suppose we are starting a Scrabble game with a rack of 7 different letters (tiles). In how many different ways might we form a 7-letter word (sequence of letters, regardless of whether it is a legal word)?

- We can pick the first letter to be any of the 7 tiles.
- For each choice of first letter, there are 6 choices of second letter, or 42 choices for the first 2 letters.
- Similarly, for each of these 42 choices there are 5 choices of third letter, and so on.
- Total number of choices = $7 \times 6 \times 5 \times \cdots \times 2 \times 1 = 7!$.
- In general, the orders (*permutations*) of n items is $n!$.
  - □ Inductive proof in book.

Ordered Selections

Suppose we want to begin the Scrabble game with a 4-letter word. In how many ways might we form the word from our 7 distinct tiles?

- The first letter is any of 7 tiles.
- For each choice of first letter there are 6 choices of second.
- For each choice of 1-2, there are 5 choices of third letter.
- For each choice of 1-2-3 there are 4 choices of fourth letter.

- Thus, there are $7 \times 6 \times 5 \times 4 = 840$ words of 4 letters out of 7.

- General rule: $\Pi(n, m)$, the number of ways to pick a sequence of m things out of n, is $n \times (n-1) \times (n-2) \times \cdots \times (n-m+1)$, i.e., the product of m integers from n downward.

  - An equivalent formula: $n!/(n-m)!$.

Combinations

Suppose we give up trying to make a word and want to throw 4 of our 7 tiles back in the pile.

- Order the 4 selected tiles in $7!/(7-4)! = 840$ ways as above.

- However, the same 4 tiles are selected in as many ways as we can order 4 tiles: $4! = 24$. Thus, the number of different choices of 4 tiles out of 7, ignoring the order of selection, is $7!/\bigl((7-4)!4!\bigr) = 7!/(3!4!) = 35$.

- General rule: We can choose m items out of n, ignoring order of selection, in $n!/\bigl((n-m)!m!\bigr)$ ways.

  - This function is usually written $\binom{n}{m}$ and spoken "n choose m."

A Recursive Definition for $\binom{n}{m}$

Key idea: If we want to choose m things out of n, we can either take or reject the first item.

- If we take the first, we can complete the choice by picking any $m-1$ of the remaining $n-1$.

  - We can do so in $\binom{n-1}{m-1}$ ways.

- If we reject the first item we must take any m out of the remaining $n-1$.

  - Do so in $\binom{n-1}{m}$ ways.

- Thus, we have an inductive definition of $\binom{n}{m}$:

Basis: $\binom{n}{0} = \binom{n}{n} = 1$ for all n.

- i.e., there is only one way to choose none or all of n elements.

Induction: $\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$ for $0 < m < n$.

- The induction parameter is a little tricky: technically it is $m(n-m)$, which is 0 for the basis (only) and decreases in the inductive step.

Inductive Definition = Direct Definition

- Use $c(n, m)$ for the inductively defined $\binom{n}{m}$.
- Proof is a complete induction on $m(n-m)$ that $c(n, m) = n!/\bigl((n-m)!m!\bigr)$.

Basis: If $m(n-m) = 0$ then $m = 0$ or $m = n$.

- If $m = 0$, then $n!/\bigl((n-m)!m!\bigr) = n!/n! = 1 = c(n, 0)$.
  - □ Note $0! = 1$ is the accepted definition.
- Similarly, if $m = n$, $n!/\bigl((n-m)!m!\bigr) = n!/n! = 1 = c(n, n)$.

Induction: We know $c(n, m) = c(n-1, m-1) + c(n-1, m)$ (inductive definition).

- Since the induction parameter $m(n-m)$ is less in both terms on the right than on the left, we may assume

$$c(n-1, m-1) = (n-1)!/\bigl((n-m)!(m-1)!\bigr)$$
$$c(n-1, m) = (n-1)!/\bigl((n-m-1)!m!\bigr)$$

  - □ Adding the left sides: $c(n, m)$.
  - □ Adding right sides: $n!/\bigl(n-m)!m!\bigr)$.