CS109B Notes for Lecture 5/19/95

Why Resolution?

Given how ugly search for proofs seems to be, and given that in general it takes exponential time to find a proof of a true statement of length n, it is remarkable how simple finding proofs can be when a technique called "resolution" is used.

Outline of Resolution

1. Convert the hypotheses and conclusion into a product (`AND`) of *clauses*.

   - A clause is a sum (`OR`) of literals.

2. "Resolve" pairs of clauses until a clause with no literals (which is equivalent to 0 (`FALSE`) is produced.

- Such an event signals that the hypothesis does follow from the conclusion. From the sequence of resolutions a proof can be found.

  - Thus, a successful resolution *is* a proof.

Converting an Expression to Product-of-Sums Form

1. Replace operators other than `AND`, `OR`, `NOT` by their equivalents in terms of those three operators.

   - e.g., $E \to F$ becomes $\bar{E} + F$; $E \equiv F$ becomes $(\bar{E} + F)(\bar{F} + E)$.

2. Use DeMorgan's laws and double negation to push `NOT`'s below `AND` and `OR`.

3. Use the distributive law of `OR` over `AND` to complete the job.

Example: Consider $\texttt{NOT}(pr \to s)$.

1. $\texttt{NOT}\big(\texttt{NOT}(pr) + s\big)$.

2. Push inner `NOT`: $\texttt{NOT}(\bar{p} + \bar{r} + s)$. Push outer `NOT`: $pr\bar{s}$.

3. Not needed; we already have a product of sums (of one literal each).

Example: $p\bar{q}+r$ needs only step (3). $(p+r)(\bar{q}+r)$.

The Resolution Operation

Based on the tautology $(p+q)(\bar{p}+r) \rightarrow (q+r)$.

- Match the left side, looking for two clauses that have between them some variable, say p, and its negation.

- Add to the set of clauses the `OR` of everything in either clause except p and $\bar{p}$.

Example: $(q+r+\bar{s})$ and $(r+\bar{q}+\bar{t})$ yield $(r+\bar{s}+\bar{t})$.

Example: $(p+q+r)$ and $(\bar{p}+\bar{q}+s)$ yield $(q+\bar{q}+r+s)$, but that is equivalent to 1 and therefore uninteresting.

- You don't need to "prove" `TRUE`.

Direct Use of Resolution

1. Convert the hypotheses and conclusion to product-of-sums form.

2. Starting with the hypotheses' clauses, resolve until you have proved all the conclusion's clauses.

Example: Let us prove $p \rightarrow q$ and $qr \rightarrow s$ imply $pr \rightarrow s$.

- From the first hypothesis: $(\bar{p}+q)$.

- From the second hypothesis: $(\bar{q}+\bar{r}+s)$.

- To prove, from the conclusion: $(\bar{p}+\bar{r}+s)$.

- The third follows from the first two by one resolution using q and $\bar{q}$.

Resolution Plus Contradiction

We were very lucky that time; there was only one thing to do and it was exactly right.

- A method that involves even less "guessing" in general is to negate the conclusion, convert the negated conclusion to product-of-sums form, and try to derive from that and the hypotheses a false clause, i.e., one with no literals at all.
  - Good heuristic, because it lets us favor making smaller clauses, heading toward a 0-literal clause.
  - Method is justified by the tautology $(p\bar{q} \rightarrow 0) \equiv (p \rightarrow q)$; $p =$ hypotheses, $q =$ conclusion.

Example: Again let us prove $p \rightarrow q$ and $qr \rightarrow s$ imply $pr \rightarrow s$.

- From the first hypothesis: $(\bar{p} + q)$.
- From the second hypothesis: $(\bar{q} + \bar{r} + s)$.
- From the negation of the conclusion (as per first example of these notes) the three one-literal clauses: $(p)(r)(\bar{s})$.

1)	$(\bar{p} + q)$	Hypothesis
2)	$(\bar{q} + \bar{r} + s)$	Hypothesis
3)	(p)	Conclusion
4)	(r)	Conclusion
5)	$(\bar{s})$	Conclusion
6)	(q)	(1) + (3)
7)	$(\bar{r} + s)$	(2) + (6)
8)	(s)	(4) + (7)
9)	0	(5) + (8)

Class Problem

We wish to prove that from the hypotheses $p + q$, $p \rightarrow r$, and $q \rightarrow r$ we can conclude r.

- Part 1: Convert the hypotheses and negation of the conclusion to clauses.
- Part 2: Derive 0 from these clauses. What have you actually proven?